Closed-Form Impulse Responses of Discrete-Domain Multidimensional Filters

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Abstract

It is known that useful two-dimensional (2-D) and three-dimensional (3-D) discrete-domain recursive transfer functions may be designed by applying the MD bilinear transformation to the continuous-domain transfer functions of prototype MD inductance-resistance networks. Closed-form expressions are derived for the impulse responses of these 2-D and 3-D discrete-domain filters.

1 Introduction

Prototype three-dimensional (3-D) inductance-resistance continuous- domain networks, having Laplace transform transfer functions of the form

$$T_1(s_1, s_2, s_3) = \frac{R}{R + s_1 L_1 + s_2 L_2 + s_3 L_3}\ , \tag{1}$$

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have been shown [1] to be useful for the design of 3-D discrete-domain recursive filter transfer functions by applying the triple bilinear transformation to (1). In particular, such filters can be used to selectively enhance 3-D linear trajectory (LT) and 3-D plane wave (PW) space-time signals.

The demonstrated usefulness of such filters in image processing has motivated this work, in which closed-form expressions are derived for the impulse responses, $h(\mathbf{n})$ (where the boldface $\mathbf{n}$ represents the integer m-tuple $n_1, n_2, \ldots, n_m$), of both the 2-D and 3-D LT filters.

Closed-form expressions for $h(\mathbf{n})$ are not generally available for MD filters, primarily because of the lack of a Fundamental Theorem of Algebra for multivariate polynomials[1]. Therefore, MD transfer functions cannot be expanded by the method of partial fraction expansion as in the 1-D case. However, closed-form expressions are available [2] for *purely first-order* m-D z-transform transfer functions. This result has led to a deeper understanding of the 2-D stability of transfer functions obtained via the 2-D bilinear transformation [3].

The availability of algebraic expressions for $h(\mathbf{n})$ facilitates further research on the input-output properties of LT and PW 3-D recursive filters, including issues relating to MD convolution and MD stability. The transfer functions considered here are more general than in [2]. In this letter, we present the derivation for the impulse response of the 2-D LT filter using the method of residues. The extension to the 3-D case is straightforward but very lengthy, and is given in [4].

[1] For single-variable polynomials, the Fundamental Theorem of Algebra allows any N^{th} degree single-variable polynomial to be factored into N first degree factors.

2 Review of the MD Inverse z-Transform

The MD inverse z-transform is given as

$$\begin{aligned} h(\mathbf{n}) &\equiv \mathcal{Z}^{-1}[H(\mathbf{z})] \\ &= \left(\frac{1}{2\pi \jmath}\right)^m \oint_{c_1}\oint_{c_2}\cdots\oint_{c_m} H(\mathbf{z}) z_1^{n_1-1} z_2^{n_2-1}\cdots z_m^{n_m-1} dz_1\, dz_2 \cdots dz_m \\ &= \left(\frac{1}{2\pi \jmath}\right)^m \oint_{c_1}\oint_{c_2}\cdots\oint_{c_m} F(\mathbf{z}) dz_1\, dz_2 \cdots dz_m \ , \end{aligned} \tag{2}$$

where

$$F(\mathbf{z}) \equiv H(\mathbf{z}) z_1^{n_1-1} z_2^{n_2-1}\cdots z_m^{n_m-1} \ , \tag{3}$$

and each integral is evaluated on a closed counterclockwise contour which lies within the region of convergence (ROC) of $H(\mathbf{z})$ and encircles the origin. In (2), if $H(\mathbf{z})$ represents the transfer function of the filter, then $h(\mathbf{n})$ is the impulse response of the filter.

The inverse z-transform technique described here involves the evaluation of multiple contour integrals. An advantage of such an approach is that the derivation involves only one z variable at a time. Therefore, it is necessary to factor the m-D polynomials $F(\mathbf{z})$ in (3) into 1-D polynomials in z_i, whose coefficients themselves are MD polynomials in $z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$. We denote an m-D polynomial in this form by
$F[z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m](z_i)$.

Determination of the inverse MD z-transform *directly* by contour integration as in (2) is extremely difficult in almost all cases. However, the contour integrals can be evaluated by application of Cauchy's Residue Theorem. This method involves evaluating residues at each of the poles inside each contour of integration c_m in (2). The resulting residues are then added together to evaluate the contour integral. For an m-D transfer function, the impulse response is obtained after m successive applications Cauchy's Residue Theorem. Rather than integrating, the problem is reduced to evaluating residues, which require closed-form derivative expressions. The required closed-form derivative expressions can be obtained either intuitively as in [4], or by Leibniz's Theorem.

3 The Closed-Form Impulse Response Expression for the Discrete-Domain 2-D LT Filter

The 2-D LT filter transfer function may be obtained by simply setting L_3 to zero in (1), and the discrete-domain transfer function is obtained from double bilinear transformation. The 2-D discrete-domain LT filter transfer function is

$$H_1(z_1, z_2) \;=\; \frac{R\,(z_1+1)\,(z_2+1)}{L_1(z_2+1)(z_1-1) \;+\; L_2(z_1+1)(z_2-1) \;+\; R(z_1+1)(z_2+1)}\,. \tag{4}$$

Since the ROC of (4) with respect to each z_m is an annulus extending outward from the pole farthest from the origin, each contour of integration, c_m, encircles all poles with respect to z_m.

In order to derive the inverse z-transform of (4), we first rewrite (4) in terms of the zeros of the denominator polynomial with respect to the z_1. Thus the integrand in the inverse z-transform definition of (2) is

$$F[z_2](z_1) = \frac{R\,z_1^{n_1-1}\,(z_1+1)\,z_2^{n_2-1}\,(z_2+1)}{(L_1 - L_2 + R + z_2(L_1+L_2+R))\left(\frac{-L_1-L_2+R+z_2(-L_1+L_2+R)}{(L_1-L_2+R+z_2(L_1+L_2+R))} + z_1\right)}\,. \tag{5}$$

The above function has two poles (with respect to z_1). The term $z_1^{n_1-1}$ in the numerator of (5) is a pole term only in the special case of $n_1 = 0$; that is, a pole exists at $z_1 = p_1 \equiv 0$ iff $n_1 = 0$ and $n_2 \geq 0$. Note the condition on n_2 is included since the impulse response is causal (in two dimensions).

The second pole in (5) exists at $z_1 = p_2 \equiv -\frac{-L_1-L_2+R+z_2(-L_1+L_2+R)}{(L_1-L_2+R+z_2(L_1+L_2+R))}$ (for $n_1 \geq 0, n_2 \geq 0$). Evaluating the residues at these two first order poles gives, respectively,

$$G_1(n_1, z_2) \equiv \text{Res}\,[F[z_2](z_1) \text{ at } z_1 = p_1] =$$
$$\begin{cases} \dfrac{R z_2^{n_2-1}(1+z_2)}{(-L_1+L_2+R)\left(\frac{-L_1-L_2+R}{-L_1+L_2+R}+z_2\right)}, & n_1 = 0,\ n_2 \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

and

$$G_2(n_1, z_2) \equiv \text{Res}\,[F[z_2](z_1) \text{ at } z_1 = p_2] =$$
$$\begin{cases} \dfrac{2L_1R(L_1-L_2-R)^{n_1-1}\left(\frac{L_1+L_2-R}{L_1-L_2-R}+z_2\right)^{n_1-1} z_2^{n_2-1}(1+z_2)^2}{(L_1+L_2+R)\left(\frac{L_1-L_2+R}{L_1+L_2+R}+z_2\right)^{n_1+1}}, & n_1 \geq 0,\ n_2 \geq 0 \\ 0, & \text{otherwise} \end{cases}. \tag{7}$$

According to the Residue Theorem, the sum of the residues in (6) and (7) gives the contour integral of $F[z_2](z_1)$ with respect to z_1. Applying the Residue Theorem a second time on each of the above two residues yields the contour integral with respect to z_2 and therefore yields the impulse response. Note that the functions $G_1(n_1, z_2)$ and $G_2(n_1, z_2)$ are nonzero for specific values of n_1 and n_2. These conditions on $G_1(n_1, z_2)$ and $G_2(n_1, z_2)$ also apply to the residues obtained from subsequent applications of the Residue Theorem.

In (6), the numerator term $z_2^{n_2-1}$ is a pole for the special case of $n_2 = 0$, and so a first order pole exists at $z_2 = p_3 \equiv 0$ if $n_1 = 0$ (from the conditions stated in (6)) *and* $n_2 = 0$. Applying the Residue Theorem to (6) gives

$$h_1(n_1, n_2) = \text{Res}\,[G_1(n_1, z_2) \text{ at } z_2 = p_3] =$$
$$\begin{cases} \frac{R}{-L_1-L_2+R}, & n_1 = 0, n_2 = 0 \\ 0, & \text{otherwise} \end{cases}. \tag{8}$$

Observe that (6) also has a first order pole at $z_2 = p_4 \equiv -\frac{-L_1-L_2+R}{-L_1+L_2+R}$ for $n_1 = 0$ and $n_2 \geq 0$ (same conditions as for (6)).

$$h_2(n_1,n_2) = \mathrm{Res}\left[G_1(n_1,z_2) \text{ at } z_2 = p_3\right] = \begin{cases} \frac{2L_2R(L_1+L_2-R)^{n_2-1}}{(-L_1+L_2+R)^{n_2+1}}, & n_1 = 0,\ n_2 \geq 0 \\ 0, & \text{otherwise} \end{cases}. \tag{9}$$

Following a similar approach as above for $G_1(n_1,z_2)$, $G_2(n_1,z_2)$ in (7) possesses poles at the following locations: $z_2 = p_5 \equiv -\frac{L_1+L_2-R}{L_1-L_2-R}$ if $n_1 = 0, n_2 \geq 0$; $z_2 = p_6 \equiv 0$ if $n_1 \geq 0, n_2 = 0$; and $z_2 = p_7 \equiv -\frac{L_1-L_2+R}{L_1+L_2+R}$ if $n_1 \geq 0, n_2 \geq 0$.

Applying the Residue Theorem to first order poles p_5 and p_6 yield, respectively,

$$h_3(n_1,n_2) = \mathrm{Res}\left[G_2(n_1,z_2) \text{ at } z_2 = p_5\right] = \begin{cases} -\frac{2L_2R(L_1+L_2-R)^{n_2-1}}{(-L_1+L_2+R)^{n_2+1}}, & n_1 = 0, n_2 \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

and

$$h_4(n_1,n_2) = \mathrm{Res}\left[G_2(n_1,z_2) \text{ at } z_2 = p_6\right] = \begin{cases} \frac{2L_1R(L_1+L_2-R)^{n_1-1}}{(L_1-L_2+R)^{n_1+1}}, & n_1 \geq 0, n_2 = 0 \\ 0, & \text{otherwise} \end{cases}. \tag{11}$$

Finally, (7) has a pole at $z_2 = p_7$ which has a multiplicity of (n_1+1). After some algebraic manipulation, it can be shown that

$$h_5(n_1,n_2) = \mathrm{Res}\left[G_2(n_1,z_2) \text{ at } z_2 = p_7\right] = \begin{cases} \frac{2\,L_1\,R}{n_1!}\frac{(L_1-L_2-R)^{n_1-1}}{(L_1+L_2+R)^{n_1+1}} \times \\ \left[\sum_{i=1}^{n_1}\left(\frac{n_1!}{i!\,(n_1-i)!}A^{n_2-i-1}\,B^{i-1}\,\left(\prod_{j=i}^{n_1-1} j\right)\,\left(\prod_{j=n_2-i}^{n_2-1} j\right)\right)\right. \\ +2\sum_{i=1}^{n_1}\left(\frac{n_1!}{i!\,(n_1-i)!}A^{n_2-i}\,B^{i-1}\,\left(\prod_{j=i}^{n_1-1} j\right)\,\left(\prod_{j=n_2-i+1}^{n_2} j\right)\right) \\ \left.+\sum_{i=1}^{n_1}\left(\frac{n_1!}{i!\,(n_1-i)!}A^{n_2-i+1}\,B^{i-1}\,\left(\prod_{j=i}^{n_1-1} j\right)\,\left(\prod_{j=n_2-i+2}^{n_2+1} j\right)\right)\right], & n_1 \geq 0, n_2 \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

where

$$A = \left(-\frac{L_1 - L_2 + R}{L_1 + L_2 + R}\right) \tag{13}$$

$$B = \left(\frac{4\,L_1\,L_2}{(L_1 + L_2 + R)\,(L_1 - L_2 - R)}\right). \tag{14}$$

Finally, adding together $h_1(n_1, n_2)$ through $h_5(n_1, n_2)$ above yields the closed-form impulse response expression

$$\begin{aligned} h(n_1, n_2) &= \frac{R}{-L_1 - L_2 + R}\,\delta(n_1)\,\delta(n_2) + \frac{2\,L_1\,R\,(L_1 + L_2 - R)^{n_1 - 1}}{(L_1 - L_2 + R)^{n_1 + 1}}\,\delta(n_2) \\ &+ \frac{2\,L_1\,R}{n_1!}\,\frac{(L_1 - L_2 - R)^{n_1 - 1}}{(L_1 + L_2 + R)^{n_1 + 1}} \times \\ &\left[\sum_{i=1}^{n_1}\left(\frac{n_1!}{i!\,(n_1 - i)!}A^{n_2 - i - 1}\,B^{i-1}\left(\prod_{j=i}^{n_1 - 1} j\right)\left(\prod_{j=n_2 - i}^{n_2 - 1} j\right)\right)\right. \\ &+2\sum_{i=1}^{n_1}\left(\frac{n_1!}{i!\,(n_1 - i)!}A^{n_2 - i}\,B^{i-1}\left(\prod_{j=i}^{n_1 - 1} j\right)\left(\prod_{j=n_2 - i + 1}^{n_2} j\right)\right) \\ &\left.+\sum_{i=1}^{n_1}\left(\frac{n_1!}{i!\,(n_1 - i)!}A^{n_2 - i + 1}\,B^{i-1}\left(\prod_{j=i}^{n_1 - 1} j\right)\left(\prod_{j=n_2 - i + 2}^{n_2 + 1} j\right)\right)\right] \end{aligned} \tag{15}$$

where

$$A = \left(-\frac{L_1 - L_2 + R}{L_1 + L_2 + R}\right) \tag{16}$$

$$B = \left(\frac{4\,L_1\,L_2}{(L_1 + L_2 + R)\,(L_1 - L_2 - R)}\right). \tag{17}$$

This closed-form expression is the required result.

4 Conclusion

A closed-form expression for the impulse responses of the discrete-domain 2-D LT filter has been derived. The derivation can be extended to filters of higher order and dimension. However, the number of poles, and therefore the required number of residues, increases with the order of the filter. A large number of dimensions means that the expressions become increasingly complex with each application of Cauchy's Residue Theorem.

Impulse response expressions are useful in the stability analysis of MD filters [3] [5], and in the design of filters through numerical optimization [6]. Further, the impulse responses of 3-D PW filters are easily obtained by convolving the impulse responses of two 3-D LT filters. The method for deriving closed-form expressions for impulse responses of 2- and 3-D LT filters may also be extended to other MD transfer functions, and to the determination of MD residues in general.

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References

[1] L. T. Bruton and N. R. Bartley, "The Enhancement of Moving Objects in Digital Images Using Adaptive Three-Dimensional Filters", *IEEE Trans. Circuits Syst.*, Vol. CAS-33, pp. 604-612, June 1986.

[2] P. Bauer, "Asymptotic Behavior of Nonlinear M-D Digital Filters with Singularities on the Distinguished Boundary", *in Proc. IEEE Int. Symp. Circuits Syst.*, pp. 3011-3014, June 1990.

[3] D. Goodman, "Some Difficulties with the Bilinear Transform", *IEEE Trans. Circuits Syst.*, Vol. CAS-24 pp. 201-208, April 1977.

[4] D. H. Jin, "A Study of Multidimensional Linear Trajectory Filters", Master's thesis, The University of Calgary, Calgary, Alberta, 1993.

[5] G. Garibotto, "A New Stability Test for 2-D Digital Filters", *in Signal Processing: Theories and Applications*, pp. 413-416, 1980.

[6] P. Agathoklis, L. T. Bruton, and N. R. Bartley, "The Elimination of Spikes in the Magnitude Frequency Response of 2-D Discrete Filters by Increasing the Stability Margin", *IEEE Trans. Circuits Syst.*, Vol. CAS-32, pp. 451-458, May 1985.