# On the Practical BIBO Stability of Multidimensional Filters

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*Abstract* - **The conventional definition of multidimensional (*m*-D) bounded-input bounded-output (BIBO) stability permits the *m*-D domain of the input signal, and of the *m*-D computations, to be of infinite extent in all m dimensions. On the other hand, the definition of *m*-D practical-BIBO (P-BIBO) stability restricts the *m*-D domain of the input signal, and of the computations, to be of finite extent in all but one of the m dimensions. It is shown that the P-BIBO stability criterion corresponds to those computations that can be made with a machine having a finite number of (infinite-precision) memory storage elements, whereas the BIBO stability criterion, which is more restrictive, also includes machines having an infinite number of (infinite-precision) memory storage elements. It is argued that the extension of the BIBO stability criterion to include such infinitely large machines is of limited practical value, and that P-BIBO stability is the more useful criterion. It is also more easy to test.**

## I. Introduction

It is well documented that, for $m > 1$, recursive *m*-D filters are considerably more difficult to design than are 1-D filters [5][7]. A principal difficulty relates to the stability of the filter algorithm. There is now a significant body of literature that has tackled the stability problem by employing an *m*-D bounded-input bounded-output (BIBO) stability criterion [3] [8] [11][13], which is essentially obtained by direct extension of the 1-D BIBO stability criterion.

We first review the concept of *m*-D BIBO stability with respect to an input signal $u(\boldsymbol{n})$ and corresponding output signal $y(\boldsymbol{n})$, related according to the *m*-D convolution sum

$$y(\boldsymbol{n}) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} u(k_1, \ldots k_m)\, h(n_1 - k_1, \ldots, n_m - k_m), \quad \textbf{(1)}$$

where $\boldsymbol{n} = n_1, n_2, \ldots, n_m$, and $h(\boldsymbol{n})$ is the *m*-D unit impulse response of a filter having positive hyper-quadrant support. With the assumption that the above recursively-computable convolution is carried out $\forall \boldsymbol{n} \in Z_+^m$, where $Z_+^m = \{ (n_1, n_2, \ldots, n_m) \mid 1 \le n_k \le \infty, 1 \le k \le m \}$, the filter $h(\boldsymbol{n})$ is said to be BIBO stable if and only if (iff), for all $u(\boldsymbol{n})$ satisfying

$$|u(\boldsymbol{n})| \le M < \infty, \ \forall \boldsymbol{n} \in Z_+^m, \quad \textbf{(2)}$$

the output $y(\boldsymbol{n})$ satisfies, for some *P*,

$$|y(\boldsymbol{n})| \le P < \infty, \ \forall \boldsymbol{n} \in Z_+^m. \quad \textbf{(3)}$$

Note that this definition demands the computation of $y(\boldsymbol{n})$ in the regions of $R^m$ where $n_1, n_2, \ldots, n_m$, are *all* infinite. The criterion for P-BIBO stability relaxes this requirement [4], requiring only that the computation be carried out in each of the regions

$$Z_+^{m/j} = \{ (n_1, n_2, \ldots, n_m) \mid (1 \le n_j \le \infty, j \in \{1, 2, \ldots, m\}), \ (1 \le n_k < \infty, 1 \le k \le m, k \ne j) \}, \quad \textbf{(4)}$$

where $j = 1, 2, \ldots, m$. In words, the regions defined by $Z_+^{m/j}$ are in the *m*-D positive hyper-quadrant, but are of finite extent in all but the $j^{\text{th}}$ dimension.

The corresponding conditions on $h(\boldsymbol{n})$ for BIBO and P-BIBO stability, are respectively,

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} |h(\boldsymbol{n})| < \infty, \textbf{ - BIBO} \quad \textbf{(5)}$$

$$\sum_{n_1=0}^{N_1} \cdots \sum_{n_k=0}^{\infty} \cdots \sum_{n_m=0}^{N_m} |h(\boldsymbol{n})| < \infty, \textit{for } k = 1, 2, \ldots, m, \textbf{ - P-BIBO} \quad \textbf{(6)}$$

where $N_j$, $1 \le j \le m, j \ne k$, are any finite integers [4].

Many researchers have also pursued the corresponding necessary and sufficient conditions in the *m*-D $\boldsymbol{z}$-domain. With respect to BIBO stability, these conditions have been difficult to establish largely because of the possibility that

$$\boldsymbol{Z}[h(\boldsymbol{n})] \equiv H(z) = \frac{P(z)}{Q(z)}, \quad (P(z) \text{ and } Q(z) \text{ irreducible}) \quad \textbf{(7)}$$

may have a non-essential singularity of the second kind (NSSK) at some $z_a$, implying that $P(z_a) = Q(z_a) = 0$, and that $H(z)$ is non-analytic and indeterminate at $z_a$ [9][10].

It has been shown that BIBO stable filters may be designed by *m*-D bilinear transformation from **s**-domain prototype transfer functions [7][8][10], $T(\boldsymbol{s}) = N(\boldsymbol{s})/D(\boldsymbol{s})$, having strictly-Hurwitz denominators, $D(\boldsymbol{s})$, and that, in such

cases, any NSSKs are isolated and on the distinguished boundary, $T^m$, of the closed $m$-D unit polydisk, $\overline{U}^m$, where

$$\overline{U}^m = \{ (z_1, z_2, \ldots, z_m) \,|\, |z_i| \leq 1 \}, i = 1, 2, \ldots m, \text{ and}$$

$$T^m = \{ (z_1, z_2, \ldots, z_m) \,|\, |z_i| = 1, i = 1, 2, \ldots m \} .$$

If NSSKs of this type *are* present, then a detailed consideration of the behaviour of $H(z_a)$, and its derivatives around $z_a$, permits the necessary and sufficient conditions for BIBO stability to be tested in terms of resultants [9]. However, the necessary and sufficient conditions are not known in terms of the filter coefficients.

A sufficient condition on $H(z)$ for BIBO stability is well known [11][12] and is given by

$$Q(z) \neq 0, \forall z \in \overline{U}^m, \quad \textbf{(8)}$$

which, while simple to state, is not easily tested in terms of the coefficient values of the difference equation. On the other hand, the necessary and sufficient condition on $H(z)$ for P-BIBO stability is given by

$$Q(0, 0, \ldots, z_i, \ldots, 0) \neq 0 \text{ for } z_i \in \overline{U}^1, i = 1, 2, \ldots, m, \quad \textbf{(9)}$$

which is simply a set of conditions on 1-D polynomials [4]. Note that (9) ensures all bounded-amplitude inputs $x(\boldsymbol{n})$, $\boldsymbol{n} \in Z_+^{m/j}$, will lead to bounded-amplitude outputs $y(\boldsymbol{n})$ for *any j*; that is, for any dimension in which the computation proceeds with infinite extent. Note that the conditions given in (9) are easily tested using widely-available 1-D polynomial root solving techniques and coefficient testing methods.

Clearly then, testing for $m$-D P-BIBO stability is very straightforward, relative to the case of BIBO stability. It has further been argued that the conditions under which P-BIBO stability apply (when $\boldsymbol{n} \in Z_+^{m/j}$) are in correspondence with the practical conditions used in image processing [4][6]. The argument is essentially that practical recursion strategies require that the domain of the recursion be of finite-extent in at least $(m-1)$ of the dimensions. For example, in the case $m = 2$ and employing row-by-row recursion, the recursion must eventually proceed from the first row to the second row, implying that the computation is of finite extent in the first row, say $n_1 = 1$. Similarly, it must be bounded in all rows to some upper number, say $N_1$. Extending and generalizing this argument to dimensions $m > 2$ and to arbitrary recursive hyper-planes, leads to the conclusion that, in general, the computation cannot be unbounded in more than one dimension.

In this contribution, we further extend the case for the relevance of the P-BIBO stability criterion by considering *the types of machines* for which P-BIBO stability is applicable.

## II. Computation of the Filter Using a Finite Number of Variables

We will use a computational model which idealizes some of the constraints typically imposed in practical filtering applications. In 1-D design, it is generally assumed that a machine is available which can store a finite number of filter coefficients and a finite number of input values *exactly*; that is, to infinite precision. The consideration of finite word lengths and truncation effects are often ignored, or at least delayed until the implementation phase. We will also use this ideal model of computation; that is, we will assume that we have a machine available which carries out all operations to infinite precision, but which may only have a *finite number* of such values available at any instant during the computation of the filter algorithm. It will be further assumed that arithmetic operations on these infinite precision values can be performed in a finite amount of time. We will denote this model as an *infinite-precision discrete-domain processor* (*IDDP*).

Note that we take care to restrict the number of variables to a finite number, rather than allowing the number to become infinite. As we shall see, this restriction has a significant impact on the types of signals which can be processed, both ideally and in practice.

The following fact is now required:

*Fact 1: If the domain of the input signal, $u(\boldsymbol{n})$, of a recursively computable filter having first hyper-quadrant support is such that $\boldsymbol{n} \in Z_+^{m/j}$, and the index of the unbounded dimension is m (m being arbitrarily selected), then there exists a finite length computational procedure such that, for a fixed $n_m$, the finite set of output values $y(\{1, \ldots, N_1\}, \ldots, \{1, \ldots, N_{m-1}\}, n_m)$ can be determined using an IDDP.*

*If more than one domain variable is unbounded, then the number of outputs $y(\{\bullet\}, \ldots, \{\bullet\}, n_m)$ will be become infinite, and hence the computational procedure would not complete in a finite amount of time. This is true regardless of the selected direction of recursion.* ♦

Fact 1 implies that, so long as only one domain variable is unbounded, a direction of recursion can be selected for which there exists a finite number of output values for each $n_m$. If we think of the direction of recursion as specifying a series of hyper-planes in $m$-D space, then Fact 1 states that the normals of these hyper-planes can always be selected to point along the direction of the unbounded index. This ensures that the hyper-surfaces defined by the intersection of these hyper-planes with the hyper-volume defined by the input domain, are finite. Fact 1 further implies that if two or more indices

become unbounded, then the area defined by these intersections becomes infinite.

Fact 1 is now used to prove the following:

*Theorem 1 : A recursively computable filter having a unit impulse response $h(\boldsymbol{n})$ with first hyper-quadrant support and implemented using an IDDP, must have an input signal, $u(\boldsymbol{n})$, such that its domain n, and the domain of the recursive computation, satisfy $\boldsymbol{n} \in Z_+^{m/j}$.*

*Proof:* Without loss of generality, assume that the index of the dimension of infinite extent is *m*. Denote the infinite sequence of input and output values associated with a particular subindex $(n_1, n_2, \ldots, n_{m-1})$ by

$$u_{n_1, n_2, \ldots, n_{m-1}}(n_m), \text{ and } y_{n_1, n_2, \ldots, n_{m-1}}(n_m), \; 1 \le n_m \le \infty, \quad \textbf{(10)}$$

respectively. Since dimensions 1 through *m*-1 are of finite extent, there are a finite number $(N_1 \times N_2 \times \ldots \times N_{m-1})$ of sequences described by (10), for a fixed $n_m$.

By assumption, the input and output masks of the filter have finite regions of support and thus only a finite number of previous input and output values are required in order to compute $y_{n_1, n_2, \ldots, n_{m-1}}(n_m)$, for a fixed $n_m$. Fact 1 ensures that, since $\boldsymbol{n} \in Z_+^{m/j}$, there exists a procedure which allows us to compute the finite set of outputs $y(\{1, \ldots, N_1\}, \ldots, \{1, \ldots, N_{m-1}\}, n_m)$.

Combining these two facts we conclude that it is possible to compute $y(\{1, \ldots, N_1\}, \ldots, \{1, \ldots, N_{m-1}\}, n_m)$ using a finite number of variables. Since $n_m$ is arbitrary, all $y_{n_1, n_2, \ldots, n_{m-1}}(n_m)$ for $1 \le n_m \le \infty$ can thus be computed using a finite number of variables. Thus, the sufficiency condition of Theorem 1 is satisfied.

In order to show the necessity that $\boldsymbol{n} \in Z_+^{m/j}$, we again use Fact 1. If two or more indices become infinite in extent then there will always be a value of $n_m$ (possibly infinite) such that there are an infinite number of outputs. Since the filter is recursive, each of these values will be required at least once when computing the next set of outputs, that is, at $n_m + 1$, and thus will require storage. Clearly, such a signal cannot be processed using an *IDDP*. Hence, only input signals whose domain is such that $\boldsymbol{n} \in Z_+^{m/j}$ are computable using an *IDDP*.♦

The relationship between Theorem 1 and P-BIBO stability is now clear. *To determine if an m-D filter, implemented using an IDDP, exhibits bounded-input bounded-output behaviour, it is sufficient that it satisfy the P-BIBO stability criterion.* More significantly, Theorem 1 implies *that an assumption underlying BIBO stability is the requirement that an infinite number of variables be available during computation.* From a practical perspective this is unrealistic. The consequence is that BIBO stability tests are too restrictive for the testing of practical filters.

The relationship between an *IDDP* and those processors which are used in practical systems is obvious: we replace the infinite memory storage required for each *IDDP* variable with a finite length register in the practical filter. In doing so, all variables including the filter output, $y(\boldsymbol{n})$, will always be bounded by the maximum and minimum value representable within these registers. However, if the filter is P-BIBO unstable, then we know that, regardless of the length of the registers, there exists an input, $u(\boldsymbol{n})$, which will cause the output, $y(\boldsymbol{n})$, to overflow. If the practical filter is P-BIBO stable, then we can ensure that such overflows cannot occur by employing registers of sufficiently long wordlength.

Of course, there remain a number of important design criterion, in addition to P-BIBO stability, that must often be considered and which are not specifically addressed here. For example, the effects of underflow and corresponding induced granularity limit cycles, of overflow and corresponding overflow-truncation induced limit cycles, dynamic range and coefficient sensitivities, are all important issues that address the ultimate robustness of the *m*-D filter implementation. Most researchers have restricted such considerations to the class of BIBO stable *m*-D filters, with perhaps one notable and recent exception in which Bauer [1][2] interestingly concludes that BIBO-unstable *2*-D filters, including those having NSSK on $T^2$, can be asymptotically-stable for finite-extent signals. This conclusion is entirely consistent with [4], and with this work. Bauer's results thus furnish theoretical evidence to support earlier work reported by Bruton and Bartley [6], where a P-BIBO stable, BIBO unstable, 2-D fan filter was used to process seismic data in a useful fashion.

## III. Conclusion

We have shown that an underlying assumption within the definition of BIBO stability is that an infinite number of variables are available for computation of the input-output difference equation of the filter. By restricting this number to a finite value, as required in practice, we have shown that the P-BIBO stability test is sufficient to ensure bounded-input bounded-output behaviour of the *IDDP* filter. This result suggests that BIBO stability is unnecessarily restrictive for practical (finite-memory) filters, and that P-BIBO stability is a more relevant criterion.

## IV. REFERENCES


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