CHAPTER 1 An Introduction To Signals and Functions

1.1 INTRODUCTION

This chapter provides a brief summary of the concepts that are employed in later chapters. The student who is familiar with the concepts of real and complex numbers, the domain, range, continuity and differentiability of functions, the idea of a complex-domain analytic function, the singularities of complex-domain functions and the regions of convergence of these functions may choose to skip this chapter.

1.2 NUMBERS

Signals are often represented by numbers and signal processing is therefore primarily concerned with mathematical operations on numbers. In the following, we define the various types of numbers.

The Natural Numbers

The most elementary set of numbers is the infinite set of non-negative integers $\{0,1,2,3,\ldots\infty\}$ where we employ the curly braces $\{\}$ to enclose the elements of a set. The non-negative integers are defined as the natural numbers and the infinite set of natural numbers is written as $\mathbb{N}$ or $\mathbb{N}^1$ where the superscript 1 implies that the set of numbers is one-dimensional; a concept that will become more clear when we shortly define multidimensional numbers.

The symbol ∞ is not a number. It is a symbol that is used to indicate that a sequence, or set of elements, extends indefinitely in the indicated direction. Thus, the sequence $\{0,1,2,3,\ldots\infty\}$ implies that the sequence continues without limit; that is, for any *arbitrarily large* positive integer *n* in the set there exists the next largest integer *n+1* in the set and we say that there is an infinite number of elements in the sequence. We will usually drop the infinity symbol if its implication in an unbounded sequence is obvious.

Entire Integers

The set $\{-\infty,\ldots,-3,-2,-1,0,1,2,3,\ldots\infty\}$ of all integers, including the negative integers, is denoted $\mathbb{Z}$ or $\mathbb{Z}^1$ and referred to as the set of entire integers. Clearly this set also contains an infinite number of elements.

If all elements of a set S_1 are contained in a set S_2, then we write this fact as $S_1 \subset S_2$. Therefore, $\mathbb{N} \subset \mathbb{Z}$.

Rational Numbers

Assuming $S_1 \subset S_2$, the removal of the set S_1 of elements from the set S_2 of elements is written S_1 / S_2. Thus, $q \in \mathbb{Z}/\{0\}$ means that q is a non-zero entire integer.

The infinite set $\mathbb{Q}$, or $\mathbb{Q}^1$, of rational numbers is the set of all possible numbers that can be formed from the ratio

$$\frac{p}{q} \tag{1.1}$$

such that $q \in \mathbb{Z}/\{0\}$ and $p \in \mathbb{Z}$.

The number

$$\frac{-77}{13} \in \mathbb{Q}^1 \tag{1.2}$$

It may be shown that the decimal part of the decimal representation of a rational number must eventually recur. For example, in the above, $-77/13 = \text{-}5.923076923076\ldots = \text{-}5.\overline{923076}$.

Clearly, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

Digital machines are able to *exactly* represent rational numbers.

Irrational Numbers

There is an infinite set of numbers that cannot be represented by employing ratios of entire integers; they are the set of so-called irrational numbers. For example, the ratio p of the circumference of a circle to its diameter is an irrational number; so is $\sqrt{2}$. Such numbers cannot be represented in decimal form *except by employing an infinite number of digits after the decimal point having no recurring part*. For example, the irrational number $p = 3.14257$...... cannot be written as a rational number because there is no repeating pattern of digits after the decimal point. Similarly, $1.4\overline{5}\overline{17} = 1.4517517517\ldots \notin \mathbb{Q}$ where the digits 517 recur without limit. Conversely, the number 1.23 is rational $(1.23 = 123/100 \in \mathbb{Q})$ and can be written $1.23\overline{0}$ where the 0 digit recurs without limit. Digital machines cannot exactly represent irrational numbers.

Real Numbers

A number, of any of the above types, is a 1D real number and *can be placed in one-to-one correspondence with a position on a straight line of infinite length*, with positive numbers extending in one direction and negative numbers in the other direction. We refer to this set of real numbers as $\mathbb{R}$ or $\mathbb{R}^1$. A real number is either rational or irrational. Therefore, the set of irrational numbers is simply the set $\mathbb{R}/\mathbb{Q}$.

It turns out to be of considerable practical importance that the irrational numbers cannot be exactly represented in a digital machine; for example, in a computer. If we plan on using irrational numbers, and we often cannot avoid such use in signal processing, then our digital machines can only approximate such numbers and, in doing so, make errors.

Multidimensional (MD) Real Numbers

The reader might expect that a textbook about MD signal processing will make use of MD numbers. In fact, the concept of a MD number is already familiar from the notion of a coordinate in a Euclidean plane. The point in the plane that is horizontal distance 2 and vertical distance –5 from the origin has coordinate $(2.12, -5\frac{1}{3})$ and is referred to as a 2-tuple or two-dimensional number. The set of all such pairs of 1D real numbers is defined as $\mathbb{R}^2$ and defines the 2D Euclidean plane. The real 2-tuple, or 2D real number, consists of two *ordered* 1D real numbers.

We may represent as a 2D Euclidian plane, of infinite extent in all directions, as shown in Figure (2.1). Each number in $\mathbb{R}^2$ may be placed in one-to-one correspondence with a point in the 2D Euclidean plane and the set of all of the numbers in $\mathbb{R}^2$ cover every point in the plane.

FIGURE 2.1 The 2-tuple Plane and the 1-tuple Plane

In general, an N-dimensional real number is an N-tuple and is any set of N ordered real numbers. Thus, we write the following 4-tuple, or 4D number, as

$$\begin{pmatrix} p & \frac{3}{4} & -1.\overline{3} & 2 \end{pmatrix} \in \mathbb{R}^4 \tag{1.3}$$

Imaginary Numbers and Complex Numbers

It is often necessary in mathematics, and this is certainly the case in signal processing, to find the square root of a negative number. Using the axiom that the product of two negative numbers is a positive number, along with the definition of the square root operation, implies that the square root of a negative number does not exist among the reals. For example, $\sqrt{-4}$ can at best be factored and written as

$$\sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{-1}\sqrt{4} = \sqrt{-1}(\pm 2) \tag{1.4}$$

Lacking a real solution to $\sqrt{-1}$ we define

$$\boxed{j \equiv \sqrt{-1}}, \tag{1.5}$$

implying that, by definition of the square root operation,

$$j^2 = -1 \tag{1.6}$$

The notation in (1.5) allows us to write, in (1.4),

$$\sqrt{-4} = \pm j2 \tag{1.7}$$

It follows that the square root of any negative real number is the product of a real number with the number j. Such numbers are defined as imaginary numbers. Equivalently, the infinite set **I** of imaginary numbers is given by multiplying the each element of the infinite set of real numbers by j.

(1.8)

The sum of a 1D real number and a 1D imaginary number, such as $(3.4+jp)$, is a 1D **complex number** and the infinite set of 1D complex numbers is written $\mathbb{C}$ or $\mathbb{C}^1$. The reader should have some familiarity with the arithmetic, algebra and properties of complex numbers.

Complex numbers may be subjected to all of the standard rules of arithmetic that apply to real numbers. Consequently, wherever the product term *(j)(j)* appears, as a result of applying the standard rules of arithmetic, it may be replaced by the real number -1.

Clearly, if the imaginary part v of a 1D complex number $u+jv$, $a,b\in\mathbb{R}$, is zero then $u+jv$ is a real number; therefore the set of real numbers belongs to the set of complex numbers; that is, $\mathbb{R}\subset\mathbb{C}$.

It is usual to represent the set of 1D complex numbers in the so-called **complex plane**, as shown in Figure (2.1). This plane is of *infinite extent in all directions* because both the real and imaginary parts of complex numbers may lie anywhere on a line of infinite length.

Writing $z=u+jv$, where u and v are the real and imaginary parts of z, we define the magnitude $|z|$ of a complex number z as

$$|z|\equiv\sqrt{u^2+v^2} \qquad \text{Magnitude} \tag{1.9}$$

which is the distance of the 1-tuple $z=u+jv$ from the origin in the complex lane. Therefore, $|z|$ is simply the distance M in Figure (2.1).

The argument $\angle[z]$ of a complex number z is defined as

$$\angle[z]\equiv\tan^{-1}(\frac{v}{u}) \tag{1.10}$$

and is simply the angle q in Figure (2.1). Clearly, the complex number z may be described by the two parameters M and q. We adopt the notation

$$z=M\angle q \tag{1.11}$$

and refer to equation (1.11) as the **polar representation** of a complex number.

Note that the complex plane $\mathbb{C}$ is not the same as the real Euclidean plane $\mathbb{R}^2$; $\mathbb{C}$ contains elements that are complex 1-tuples (such as $2+j3$), whereas $\mathbb{R}^2$ contains numbers that are real 2-tuples, such as (such as $(2,3)$).

Multidimensional Complex Numbers

Note that multidimensional complex numbers are defined by direct extension of the real case. Thus, the complex 2-tuple

$$\left((2+j5),(3-j8)\right) \tag{1.12}$$

belongs to the infinite set of complex 2-tuples and we refer to this set as $\mathbb{C}^2$. In general, the n-dimensional complex *N*-tuple belongs to $\mathbb{C}^N$.

Note that a complex *N*-tuple is not the same thing as a vector. An N-tuple is simply a number or point in $\mathbb{C}^N$ whereas a vector has both magnitude and direction.

Geometric interpretations of multidimensional complex numbers are not easily visualized by most humans ! The number in (1.12) can be thought of as existing at a point in a 4D plane for which there are two real and two imaginary axes but our ability to imagine a space in which all four axes are mutually orthogonal is beyond us. For this reason, geometric visualization will give way to algebra when representing regions in $\mathbb{C}^N$, $N>1$.

1.2.1 Intervals and Regions

We refer to intervals on the real line $\mathbb{R}^1$, areas in $\mathbb{R}^2$, volumes in $\mathbb{R}^3$ and other regions in higher dimensions.

The closed interval [*a; b*] on the real line is the set of real numbers x satisfying the relation $a \le x \le b$, where it is implied that $a \le b$. Note that the boundary points of the interval, a and b, are included in the interval. We refer to a region that includes its boundary points as being **closed**. Otherwise the region is said to be **open** and does not include its boundary points. The corresponding open interval is denoted (*a ; b*) and is therefore the set of real numbers x satisfying the relation $a<x<b$.

We identify **2D rectangles** and **3D solid-rectangles**. In the 3D case, the *closed* solid-rectangle is given by

$$\mathbf{R}^{\mathbf{3}}_{\mathbf{rec}} = \{x = (x1, x2, x3) \,|\, (x1 \in [a;b]), (x2 \in [c;d]), (x3 \in [e;f])\} \qquad (1.13)$$

which we write, for brevity, in terms of its opposite vertices as $\mathbf{R}^{\mathbf{3}}_{\mathbf{rec}} = [(a,c,e),(b,d,f)]$. A similar expression applies for the 2D rectangle.

Circles and Discs

The circle of unit radius in $\mathbb{R}^2$ with centre at the origin $(0,0)$ is denoted O^2 and is therefore defined as

$$O^2 \equiv \{x = (x1, x2) \in \mathbb{R}^2 \mid x1^2 + x2^2 = 1\} \qquad (1.14)$$

However, the circle in $\mathbb{C}^1$ with centre at the origin $(0,0)$ is denoted T^1 and is therefore defined as

$$T^2 \equiv \{z = (u + jv) \in \mathbb{C}^1 \mid u^2 + v^2 = 1\} \qquad (1.15)$$

and is shown in Figure (2.2). The distance of this complex number z from the origin is the magnitude of that complex 1-tuple number $|z|$ so we may write relation (1.15) as

$$T^1 = \{z \,|\, |z| = 1\} \qquad (1.16)$$

FIGURE 2.2 The Unit Circle and the Closed Unit Disc in the 1D Complex Plane

Similarly, the 2D closed unit disc $\bar{U}^1$ in $\mathbb{C}^1$ is defined as

$$\bar{U}^1 \equiv \{z \mid |z| \leq 1\} \tag{1.17}$$

where the bar over $\bar{U}^1$ implies that the region is closed; that is, it includes the boundary region in equation (1.16). The open unit disc U^1 in $\mathbb{C}^1$ is given by

$$U^1 \equiv \{z \mid |z| = 1\} \tag{1.18}$$

and therefore

$$\bar{U}^2 = U^2 + T^2 \tag{1.19}$$

where the plus sign means the union of the elements in the two sets.

Polycircles and Polydiscs

For higher dimensions, we refer to the circles and discs regions as polycircles and polydiscs. Some care must be taken when extending well known shapes from 2D or 3D to higher dimensions.

For example, the N-dimensional unit circle, or unit polycircle, in $\mathbb{R}^N$ is the region

$$T^N \equiv \{x \mid x_1^2 + x_2^2 + \ldots x_N^2 = 1\} \tag{1.20}$$

and is the set of all real N-tuples that are unity 'distance' from the origin. For $N = 3$, this describes all 3-tuples on the surface of the unit-radius sphere having its centre at the origin in 3D Euclidean space. Although the 4D sphere is evident from (1.20), the

4D geometry must rest with the algebra and not with our ability to imagine the shape of the spherical surface in 4D.

The **unit polycircle** in $\mathbb{C}^N$ is given, in terms of the N-tuple $\mathbf{z} \equiv (z_1, z_2, \ldots z_N)$, by

$$T^N \equiv \{\mathbf{z} \mid \bigcap_{i=1}^{N} |z_1| = 1\} \tag{1.21}$$

where $\bigcap_{i=1}^{N} |z_i| = 1$ means the set of complex N-tuples where $|z_1| = 1$ and $|z_2| = 1$ and $\ldots$ $|z_N| = 1$. Note that this definition is far more restrictive than if we had chosen the definition of T^N to correspond to the much larger region where $|z_1^2 + z_2^2 + \ldots z_N^2| = 1$.

The **closed unit polydisc** in $\mathbb{C}^N$ is given by

$$\bar{U}^N \equiv \{\mathbf{z} \mid \bigcap_{i=1}^{N} |z_1| \leq 1\} \tag{1.22}$$

Note that, if $|z_i| > 1$ for *any* $i = 1,2,\ldots N$ then the corresponding complex N-tuple does not belong to $\bar{U}^N$ no matter how small the magnitudes of the remaining z_i.

1.3 ONE DIMENSIONAL SIGNALS OR FUNCTIONS

A one dimensional (1D) signal may be decribed mathematically as a function of *one* independent variable. The signal may be written $x(t)$ where t is the independent variable. We use t to represent a real independent variable and z to represent a complex independent variable. In the latter case, the signal or function is written $x(z)$.

1.3.1 The Domain, Extent and Region of Support of 1D Signals

Domain

The domain of a 1D signal $x(t)$, abbreviated as $dmn[\, x(t)\,]$, is the set of t over which the signal $x(t)$ is defined. The range of a signal function , abbreviated $rng[\, x(t)\,]$, is the set of values of that correspond to $dmn[\, x(t)\,]$.

Example 1

The real signal function , shown in Figure (2.3), has domain $dmn[\,x(t)\,] = [a;\,b]$ and $rng[\,x(t)\,] = [c\,;d]$, where $a,b,c,d \in \mathbb{R}^1$..

FIGURE 2.3 Domain and Range of a 1D Signal

Continuous Domain Signals

Consider 1D signals $x(t)$ for which $dmn[\,x(t)\,] = [a;b] \in \mathbb{R}^1$. Then $x(t)$ is a continuous-domain signal if it is defined on a continuum over the closed interval $[a;b]$. A more rigorous definition requires that, for each point t_0 *inside* the interval $[a;b]$, the signal $x(t)$ is defined at all points arbitrarily close to t_0 . The signal in Figure (2.3) is an example of a continuous-domain signal.

The above definition extends to the case of complex-domain signals $x(z)$, in which case the domain $D \in C^1$ and the signal $x(z)$ must be defined at every complex number in the domain D .

Discrete-Domain Signals

Let the real domain of a 1D signal be a (possibly infinite) set of numbers $\{\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots\}$. This is equivalent to stating that the signal is defined only at discrete points $t_k, k \in \mathbb{Z}^1$. We refer to such a signal as a discrete-domain signal. If the real variable t is time, then the signal is referred to as a discrete-time domain signal. We usually assume that the numbers t_k are ordered; that is, $t_k < t_{k+1}, \forall k \in \mathbb{Z}^1$.
.

As a matter of notation, we should strictly write the signal as a discrete-domain **sequence** $\{x(t_k)\}$. However, the complete sequence is usually implied when we write one element of the sequence without the curly brackets as $x(t_k)$.

If

$$t_{k+1} - t_k = T\ , \quad \forall k \tag{1.23}$$

where T is a real positive constant, then we write the signal as $x(kT)$ which is a **uniformly-sampled** discrete-time signal. Note that $x(kT)$ is only defined at integer multiples of T There exists an infinite number of continuous-time signals $x(t)$, $t \in \mathbb{R}^1$, that yield a given discrete-time signal $x(kT)$ after sampling at integer multiples of T .

The Region of Support (ROS) of a Signal

Suppose a signal is defined to equal zero over regions of its domain D and let all such regions be denoted D_0. Then the signal is said to lack support in D_0 and its ROS is the remaining region of its domain D/D_0.

For example, suppose a 1D continuous-domain signal $x(t)$ has domain $\mathbb{R}^1$ and $x(t) \equiv 0 \ \ \forall t < 0$ and is not defined to equal zero for $t \geq 0$. Then, the ROS of the signal $x(t)$ is the part of its domain given by $t \geq 0$. We say that such a signal is **positive-sided**.

The Extent of a Signal

The extent of a signal refers to its ROS.

For the case of real-domain continuous-domain signals, if the ROS extends to either or both $+\infty$ and $-\infty$, then the signal is said to be of **infinite extent**. Otherwise, it is said to be of **finite extent**. For such a finite-extent signal $x(t)$, there exist finite numbers t_U and t_L, with $t_U > t_L$, such that

$$x(t) \equiv \begin{matrix} 0, & \forall t < t_L \\ 0, & \forall t > t_U \end{matrix} \tag{1.24}$$

If t is time, it is common for the word *extent* to be replaced by the word *duration* We say that a finite-extent time-domain signal is **duration-bounded** and that an infinite-extent time-domain signal is **duration-unbounded**.

1.3.2 The Value of Signals and Functions

Real-, Complex- and Integer-Valued Signals

If there exists some t in $dmn[\,x(t)\,]$ such that *the value* of $x(t)$ is a complex number, then $x(t) \in \mathbb{C}^1$ and the signal $x(t)$ is a *complex-valued signal* . Usually, the word *value* is dropped and we say that the signal is a **complex signal**.

If $x(t)$ is real-valued everywhere $dmn[\,x(t)\,]$ then $x(t) \in \mathbb{R}^1$ and the signal $x(t)$ is a *real-valued signal* and we say that the signal is a **real signal**.

Example 2

Consider the signal

$$x(t) \equiv \begin{matrix} 3t^2, & \forall t \in [0;2] \\ 0, & \forall t \in [-5;0^-] \end{matrix} \tag{1.25}$$

Clearly, $x(t)$ is only defined over the closed interval $[-5;2]$ and therefore $dmn[\,x(t)\,]$=$[-5;2]$. The ROS[$x(t)$]=$[0;2]$ and the range of $x(t)$ is given by $rng[\,x(t)\,]$=[0;12]. Also, $x(t)$ is a real signal of finite extent.

Example 3

Consider the signal

$$x(t) = \cos(t) + j\sin(t), \quad \forall t \in \mathbb{R}^1, \quad j \equiv \sqrt{-1} \tag{1.26}$$

as shown in Figure 2.2. The signal is defined $\forall t \in \mathbb{R}^1$ and therefore $dmn[x(t)] = \mathbb{R}^1$.

FIGURE 2.4 A 1D Complex Signal

Clearly, there exists values of t such that $x(t)$ is complex-valued, corresponding to the values of t where $\sin(t) \neq 0$. Therefore, $x(t)$ is a complex signal of infinite extent. The magnitude function $|x(t)|$

and the argument function $\angle x(t)$ of the complex signal $x(t)$ are given by

$$|x(t)|^2 = \cos^2(t) + \sin^2(t) = 1 \tag{1.27}$$

$$\angle x(t) = \tan^{-1}\left(\frac{\sin(t)}{\cos(t)}\right) = t \tag{1.28}$$

implying that the $rng[x(t)] = O^1$, the circle of unit radius and centred at the origin in $\mathbb{C}^1$.

Single-Valued Functions and Signals

A function is single-valued if, for each and every defined value of the independent variable t, *x(t)* has a unique value.

Example 4

The signal

$$x(t) = \sqrt{t}\,, \quad \forall t \in \mathbb{R}^1 \tag{1.29}$$

is not single-valued because, for all non-zero t, there are two possible values for *x(t)*. For example, if $t = 4$ then $x(t) = \pm 2$. Unless stated otherwise, all signals are assumed to be single-valued functions of the independent varaible(s).

Continuous-Valued and Discontinuous-Valued Signals and Functions

We have so far considered the continuous and discrete nature of the *domain* of a signal. We now discuss the continuous or discrete nature of the *value* (and associated range) of a real-valued signal. First, we need to define the existence of a limit of a function.

Existence of the Limit at a Point:

The limit $L(t_0)$ of a function $x(t)$, at the point t_0, is defined as

$$L(t_0) \equiv \lim_{t \to t_0}[x(t)] \tag{1.30}$$

The limit $L(t_0)$ is said to exist at the point t_0 iff (if and only if) it is unique. We use the word unique to mean single-valued and finite.

Consider, for example, the case where the domain $t \in \mathbb{R}^1$. Let t_{01} and t_{02} be *arbitrarily close* to t_0 with t_{01} less than t_0 and t_{02} greater than t_0, where t_0 is in the domain of the signal . Suppose that left-hand limit $L(t_{01})$ and the right hand limit $L(t_{02})$ are *unequal*. Then the limit $L(t_0)$ is clearly not unique at t_0 because its value depends on the direction in which the limit is taken; the limit $L(t_0)$ therefore does not exist at the point t_0 .

Consider now the more general case of a function or signal having a complex domain given by $x(z)$, $z \in \mathbb{C}^1$. A point $z_0 \in \mathbb{C}^1$ can be approached *in any direction in* $\mathbb{C}^1$ when defining the limit at of a complex domain function according to

$$L(z_0) \equiv \lim_{z \to z_0}[x(z)] \tag{1.31}$$

This limit exists iff the *same* unique limit is obtained for all directions, as illustrated in Figure 2.4.

The definitions of the limit in equations (1.30) and (1.31) do not require t (or z) to take on the value t_0 (or z_0) but *only to approach arbitrarily close to* t_0 *(or* z_0*)*. Interestingly, *it is therefore possible for the limit to exist at the point* t_0 *(or* z_0*) when* t_0 *(or* z_0*) is infinite in magnitude or simply not defined.*

Continuous-Valued at a Point

For a signal or function $x(t)$ to be continuous-valued at a point t_0, the limit $L(t_0)$ must exist *and* the (unique) value of $L(t_0)$ must equal the value $x(t_0)$ of the signal or function at t_0. This is a stronger condition than simply the existence of the limit. Thus, a signal $x(t)$, $t \in \mathbb{R}^1$, is continuous-valued *at a point* t_0 in its domain

If the limit $L(t_0)$ exists and is given by

$$L(t_0) \equiv \lim_{t \to t_0}[x(t)] = x(t_0) \tag{1.32}$$

This condition ensures that $x(t)$ is continuous-valued at t_0. Such a function must therefore be defined at t_0.

A signal or function that is not continuous-valued at the point t_0 is said to have a discontinuity at t_0.

The definition in (1.32) also applies also to signals over complex domains, such as $x(z)$ at the point z_0.

Continuous-Valued Signals or Functions

Definition: $x(t)$, $t \in \mathbb{R}^1$, is **continuous-valued** in its domain if and only if it is continuous-valued at *every* point t_0 in its domain.

A signal (or function) that is not continuous-valued is said to be **discontinuous valued** in its domain, as illustrated in Figure (2.5).

FIGURE 2.5 A Discontinuous Valued 1D Real Signal

Example 5

The function $\cos(t)u(t)$, $t \in \mathbb{R}^1$, .is discontinuous-valued because its limit is not unique at $t = 0$ whereas the function $\sin(t)u(t)$ is continuous-valued because its limit is unique $\forall t \in \mathbb{R}^1$.

The above definition of continuous-valued also applies to signals $x(z)$ over complex domains z.

1.3.3 Differentiable and Analytic Signals and Functions

It is often important that we know the conditions under which signals or functions are differentiable in their domains.

Derivative At A Point

Real Domain Functions: The *derivative* of a real domain signal or function $x(t)$, $t \in \mathbb{R}^1$, at the point t_0 in its domain, is defined as follows

$$\frac{d}{dt}[x(t_0)] \equiv \lim_{t \to t_0}\left[\frac{x(t)-x(t_0)}{t-t_0}\right] \tag{1.33}$$

This function is said to be **differentiable at** t_0 iff the derivative (that is, the limit) in equation (1.33) *exists*. Equivalently, the derivative exists iff the limit in equation (1.35) *is both finite and unique*.

We state, without proof, that a function $x(t)$ that is differentiable at t_0 is also continuous-valued at t_0. *The converse is not necessarily true*. For example, the function $x(t) \equiv |\sin(t)|$, $t \in \mathbb{R}^1$, is continuous-valued but its derivatives at integer multiples of $2p$ are not unique. [For further reading, consider *Elements of Complex Variables*, Pennesi, Holt, Reinhart and Winston, 1976, p95].

Example 6

Consider the unit step signal

$$u(t) \equiv \begin{matrix} 1, & t \geq 0 \\ 0, & t < 0 \end{matrix}, \quad t \in \mathbb{R}^1 \tag{1.34}$$

(1.36) ,

This function *has a limit that does not exist at* $t = 0$ because the limit there is not unique; the left-side limit is *L(0-) = 0* and the right side limit *L(0+) = 1*. It follows that $u(t)$, as defined in (1.34), is a discontinuous-valued function having a discontinuity at $t = 0$.

It is also easy to show that the limit in equation (1.33) is not unique at $t = 0$. The left-side limit is ∞ whereas the right-side limit is 0. Therefore, the limit in equation (1.33) is not unique and consequently the function is not differentiable at the point $t = 0$ in its domain and therefore is a non-differentiable function. It is easy to show that all discontinuous-valued functions are non-differentiable.

Example 7

The function

$$x(t) \equiv |t|, \quad t \in \mathbb{R}^1 \tag{1.35}$$

is continuous-valued because its limit is finite, unique and equal to $x(t_0), \ \forall t_0 \in \mathbb{R}^1$. However, it is not differentiable because the limit in equation (1.33) is not unique at $t_0 = 0$, being equal to 1 for the right-side limit and -1 for the left-side limit about t_0 in equation (1.33).

Derivatives of Complex Domain Functions an Analytic Functions: The *derivative* of a complex domain signal or function $x(z)$, $z \in \mathbb{C}^1$, at a point z_0 in its domain is defined as

$$\frac{d}{dz}\left[x(z_0)\right] \equiv \lim_{z \to z_0}\left[\frac{x(z) - x(z_0)}{z - z_0}\right] \tag{1.36}$$

In general, the derivatives of functions over complex domains z are complex functions. Most importantly, *there is an infinite number of derivatives that exist at any point z_0 because the limit may approach from any direction in $\mathbb{C}^1$.*

A continuous-valued function $x(z)$, $z \in \mathbb{C}^1$, is said to be **analytic at the point** z_0 iff it is differentiable at the point z_0. It is said to be **analytic in its domain** if it is differentiable everywhere in its domain. If we simply say that a function is analytic, we imply that it is analytic in its domain. (Synonyms for analytic are **holomorphic, regular and monogenic**). It therefore follows that *a function is analytic iff its derivative is finite and unique throughout its domain.*

Analytic functions are a very special class of functions that are of major importance in the theory, analysis and design of linear systems. For many applications, the underlying theory requires that a complex function be analytic.

The Cauchy-Reimann conditions provide the necessary and sufficient condtions for a complex function $x(z)$ to be analytic and are expressed as follows. Writing

$$z = s + jw\,, \quad s\,, w \in \mathbb{R}^1 \tag{1.37}$$

it follows that we may write any function of a complex variable in the form

$$x(z) = x(s + jw) = u(s\,,w) + jv(s\,,w) \tag{1.38}$$

where u and v are real functions of s and w. Then, $x(z)$ is analytic if and only if

$$\boxed{\frac{\partial u}{\partial s} = \frac{\partial v}{\partial w}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial w} = \frac{\partial v}{\partial s}} \quad \text{CAUCHY-REIMANN CONDITIONS} \tag{1.39}$$

The interested reader may refer to [Sneddon, p121]. Some remarks are in order. First, that functions of a complex variable must be highly constrained in their real and imaginary parts in order to be analytic. Second, the tractable development of circuits and systems often requires that the describing complex functions be analytic.

It was also shown by Cauchy that analytic functions are repeatably differentiable everywhere in their domain; that is, all the derivatives of an analytic function

$$\frac{d^k}{dz}[x(z)], \qquad k = 1,2,3,\ldots\infty \tag{1.40}$$

exist and are therefore unique and finite.

1.3.4 Taylor's Power Series Expansion of Analytic Functions

The existence of all higher order derivatives according to equation (1.40) allowed Taylor to obtain a truly remarkable result. Given any disc of radius *R* and centre *a* in $\mathbb{C}^1$, **where that disc lies entirely in the domain of an analytic function**,

$$\boxed{x(z) = \sum_{k=0}^{\infty} \frac{(z-a)^k}{k!}\left(\frac{d^k}{dz}\left[x(z)\right]\Big|_{z=a} \right)} \tag{1.41}$$

This result is exact. It is truly remarkable because it states that we can find the value of an analytic function *anywhere in the disc* just by knowing all of its derivatives at the centre of the disc. It follows directly that equation (1.43) is an infinite series involving constant-valued increasing derivatives of $x(z)$, evaluated at the centre of the disc. *It follows from equation (1.41) that any analytic function in the disc (that lies in the analytic region) may be written as a convergent power series*

$$x(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \dots \qquad (1.42)$$

where c_k are constants. Obviously, the c_k in equation (1.42) consist of the factorial and derivative terms in equation (1.41). Often, we know that the function of interest is analytic at the origin so that we may choose $a = 0$ in the above equation, obtaining the following Taylor Power Series about the origin

$$\boxed{x(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots}, \qquad (1.43)$$

otherwise known as the Maclaurin Series.

Note that, although all of the abovementioned disc is in the domain of the analytic signal, the analytic region is usually much larger than any disc that can be found. In fact, one can place discs having many different centres a and radii R to cover the analytic region and write a *different* Taylor Series for each one, describing a *different* convergent power series that describes $x(z)$ exactly in each disc.

To summarize, the Maclaurin Power Series in equation (1.43) may be used to describe any analytic function $x(z)$ of a complex variable that is analytic at the origin $z = a$. This powerful result underpins much of the theory upon which this subject rests. It explains why polynomials of complex variables are central to so many analysis problem.

Singularities and The Fundamental Theorem of Algebra

Singularities are defined as regions in $\mathbb{C}^1$ where $x(z)$ is non-analytic and are known as **singularities** of $x(z)$. In the case of 1D functions, it can be shown that such regions are isolated points z_0, referred to as **isolated singularities**. (For the MD case, $N > 1$, singularities are not restricted to isolated N-tuples in $\mathbb{C}^N$.)

Polynomial Functions

Taylor's Series leads us to the important conclusion that analytic functions can be represented in their discs of convergence by power series that are **polynomials in the independent variable z**. We can draw some immediate and important conclusions.

First, any constant-coefficient *finite-degree* polynomial

$$P(z) = \sum_{k=0}^{N} c_k (z-a)^k \tag{1.44}$$

must be analytic throughout $\mathbb{C}^1$ because it contains a finite number N of finite-valued (differentiable) terms $c_k (z-a)^k$. Thus, **only infinite length polynomials can be non-analytic** in $\mathbb{C}^1$.

Inverse Finite Length Polynomials Functions

Let us now consider a particularly relevant class of function, which is the inverse finite-length polynomial function

$$\frac{1}{Q(z)} \tag{1.45}$$

where $Q(z)$ is a finite degree polynomial

$$Q(z) = \sum_{k=0}^{M} b_k z^k \tag{1.46}$$

Now, it is generally possible (although tedious) to carry out the long division on equation (1.46) to arrive at an equivalent **infinite power series** representation of the form

$$\frac{1}{Q(z)} = \sum_{j=0}^{\infty} d_j z^{-j} \tag{1.47}$$

We often need to know whether, and in what regions of $\mathbb{C}^1$, we can integrate or differentiate inverse polynomial functions of this type. To answer this question, we need to know where such inverse finite-degree polynomials have their singularities; that is, what values of z, given the constants b_j, correspond to the singularities of $1/Q(z)$?

The Fundamental Theorem of Algebra

Fortunately, the so-called Fundamental Theorem of Algebra (FTA) provides the answer to the above question. This theorem states that **any polynomial of finite degree M is always factorizable** in the form

$$Q(z) \equiv \sum_{k=0}^{M} b_k z^k = b_M \prod_{k=1}^{M} (z - z_k) \tag{1.48}$$

where the M numbers $z_k \in \mathbb{C}^1$, $k = 1,2,3,\ldots M$, are called the zeros of $Q(z)$ because

$$Q(z_k) = 0, \qquad k = 1,2,3,\ldots M \tag{1.49}$$

Clearly, the ratio of polynomials $P(z)/Q(z)$ equals zero at the zeros of $P(z)$ and is of infinite magnitude at the zeros of $Q(z)$. The zeros of $Q(z)$ are referred to as the **poles** of the rational function $P(z)/Q(z)$.

It may easily be shown, by partial fraction expansion and for the case of distinct zeros, that equation (1.47) may be expressed in the form

$$\frac{1}{Q(z)} = \sum_{j=1}^{M} \frac{R_j}{z - z_j}, \qquad R_j, z_j \in \mathbb{C}^1 \tag{1.50}$$

where the terms R_j are referred to as the residues of the expansion and the terms z_j as the poles of $1/Q(z)$. The individual terms

$$\frac{R_j}{z - z_j} \tag{1.51}$$

are easily shown to be analytic everywhere except at the pole where $z = z_j$ and therefore equation (1.50) is analytic everywhere except at the M poles.

It follows that **rational functions $P(z)/Q(z)$ are analytic except at the M poles**; **we may therefore integrate or differentiate such rational functions over regions of $\mathbb{C}^1$ that are devoid of poles**. This property of rational functions of a complex variable is central to the subject at hand and particularly to use of Laplace, Fourier and Z transforms.

Fortunately, widely available robust algorithms exist, in the case of 1D functions, for finding the poles.

Region of Convergence (ROC)

A ROC of a possibly-infinite power series, such as equation (1.46), is simply a region in $\mathbb{C}^1$ corresponding to the values of z where the power series converges. That is, a region where it is analytic. The disc that is used in the Taylor Series of equation(1.41) and Maclaurin Series of equation (1.43) is a ROC and must lie in the analytic region. The disc is a ROC provided that it does not contain any of the poles. In general, there is an infinite number of such discs that can be defined on $\mathbb{C}^1$ to avoid the polar points.

We also note that **finite-length polynomials** do not have poles and are analytic everywhere in $\mathbb{C}^1$.

Example 8

Consider the rational function

$$\frac{P(z)}{Q(z)} = \frac{z+1}{z-1}, \quad z \in \mathbb{C}^1 \tag{1.52}$$

Then $P(z) = z+1$ and $Q(z) = z-1$. The zero of $P(z)$ is at $-1+j0$ and the zero of $Q(z)$ is at $1+j0$ and therefore the pole of the rational function $P(z)/Q(z)$ is at $1+j0$. The pole-zero diagram is shown in Figure (2.6).

FIGURE 2.6 Pole-Zero Diagram for

The region of analyticity is the complete complex plane with the exception of the pole location at. $1+j0$

Consider now the infinite series for this rational function. We obtain, by long division,

$$Q(z)=\frac{1}{z-1}=z^{-1}(1+z^{-1}+z^{-2}+z^{-3}+\ldots\infty) \tag{1.53}$$

which is an infinite-series representation. Then, multiplying by $P(z)$ gives

$$\frac{P(z)}{Q(z)}=\frac{z+1}{z-1}=(z+1)z^{-1}(1+z^{-1}+z^{-2}+z^{-3}+\ldots\infty)=1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^{2}+\left(\frac{1}{z}\right)^{3}+\ldots\infty$$

which is in the form of the Maclaurin Series of equation (1.45) but with z replacing $1/z$. Equation (1.53) therefore has its ROC in any disc having its centre at the origin ($1/z=0+j0$) that does not enclose the pole at $(1+j0)$. Thus, the radius of the disc must lie in the interval $0\le R<1$. Therefore, the ROC of equation (1.53) is given by the disc of largest radius, as shown in Figure (2.6) , corresponding to the condition

$$\left|\frac{1}{z}\right|<1$$

or, equivalently,

$$|z|>1$$

FIGURE 2.7 Region of Convergence of in the complex variable