CHAPTER 2 An Introduction To Multidimensional Signals and Functions

2.1 INTRODUCTION

The concepts of domain, range and region of support (ROS) of a signal are extended in this chapter to the multidimensional (MD) case. Two-dimensional (2D) and three-dimensional (3D) signals are of special interest and are considered in some detail. Examples are given of practical situations. For example, the domain, range and ROS are considered for such common signals as photographic and television images.

2.2 THE DOMAINS OFMD SIGNALS

A MD signal $x(\mathbf{t})$ has a domain $dmn[x(\mathbf{t})]$ that consists of a (possibly infinite) set of N-tuples. For example, $dmn\ [x(\mathbf{t})]$ might belong to $\mathbf{R}^{\mathbf{N}}$ or, alternatively, to $\mathbf{N}^{\mathbf{N}}$.

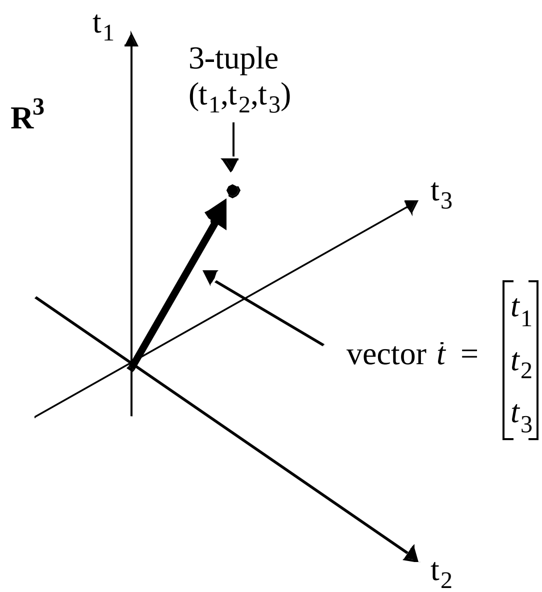
It is sometimes convenient to express the independent variable $\mathbf{t} = ((t_1)(t_2)(t_3)\ldots(t_N))$ in the form of the column matrix

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \dots \\ t_M \end{bmatrix} \tag{2.1}$$

We may also think of $\mathbf{t}$ as a vector having components $t_1, t_2, \dots, t_N$. Using the superscript $\mathbf{T}$ to indicate the operation of matrix transposition, the above equation may be written as the row matrix $\mathbf{t} = [t_1, t_2, \dots, t_d, \dots, t_N]^{\mathbf{T}}$. The N-tuple and vector representations of the independent variable are shown in Figure (2.1) for the 3D case. (Note that, if the t_d are complex, then the N-tuple representation lies in $\mathbf{C}^{\mathbf{N}}$, which is the MD complex space). For both the N-tuple representation and the vector representation of $\mathbf{t}$, the signal $x(\mathbf{t})$ is a function of N independent variables $t_1, t_2, \dots, t_d, \dots, t_N$.

The variables t_d are referred to as the **dimensions** of the signal; they are also the elements in the column matrix representation and the vector components in the vector representation of $\mathbf{t}$.

FIGURE 2.1 *N*-tuple and Vector Representations of the independent 3D variable **t**



Continuous-, Discrete- and Mixed-Domain MD Signals

Assume $\mathbf{t} \in \mathbf{R^N}$. Then the domain $dmn[x(\mathbf{t})] \in \mathbf{R^N}$. Consider now the MD case where the number of dimensions $N>1$. If $x(\boldsymbol{t})$ is defined over a continuum of N-tuple numbers in $\mathbf{R^N}$, then $x(\boldsymbol{t})$ is a continuous-domain MD signal.

If $x(\mathbf{t})$ is only defined on discrete N-tuples $\mathbf{t_k} = (t_{1k}, t_{2k}, t_{3k}, \ldots, t_{Nk}), k \in \mathbf{Z^N}$, then $x(\mathbf{t})$ is a **discrete-domain MD signal**.

Interestingly, it is possible for a MD signal to be defined over a continuum of numbers over some of its dimensions and only on discrete tuples over the remainder of its dimensions. We refer to such a signal as a **MD mixed-domain** signal. For example, the signal might be defined on the continuum of reals $\mathbf{R^1}$ over the first K dimensions and only on the entire integers $\mathbf{Z^1}$ over the remaining N-K dimensions. Then, we use the following notation for the mixed domain

$$dmn[x(\mathbf{t})] \in \mathbf{R^K x Z^{N-K}} \tag{2.2}$$

2.3 2D and 3D Image Signals

In this section, we introduce some typical 2D and 3D image signals ; that is, signals that correspond to images that may be viewed by the human vision system (HVS). Their domains and mathematical representations are discussed, as well as some of their properties. We consider, as examples, a rectangular photograph, a raster-scanned version of that photograph and digitzed images, including digitized temporal image sequences such as digital television images.

2.3.1 2D Continuous-Domain Images $x(\mathbf{t}) = x(t_1, t_2)$

Many images are perceived by the HVS to be continuous-domain images. For example, viewed by the HVS from a sufficient distance from its surface and from a direction that is normal to the surface, a typical gray-tone photograph $x(\mathbf{t}) = x(t_1, t_2)$ has a 2D rectangular domain

$$\mathbf{R^2_{rec}} = [(a;b)(c;d)] \in \mathbf{R^2} \tag{2.3}$$

where the value of $x(t_1, t_2)$ is defined here as a real number that is directly proportional to the gray-level intensity of the photographic image at the 2-tuple point (t_1, t_2) .

A typical gray-level photograph, including its black border, may be represented as a 2D continuous-domain signal, as illustrated in Figure (2.2) , having zero value where the intensity is black and unity value where the intensity is white. In this example, the rectangular domain $\mathbf{R}^2_{\mathbf{rec}}$ includes the black border *where the signal is defined to be zero*. Therefore, by definition, the *region of support* (ROS) of the signal is the smaller interior rectangle $\mathbf{R}^2_{\mathbf{ROS}}$ that does not include this zero-valued border. 2D photographic signals, and virtually all practical 2D spatial images, are clearly of finite-extent; that is, their domains are closed subsets of $\mathbf{R}^2_{\mathbf{rec}}$ having boundary points that are finite distances from all points in $\mathbf{R}^2$.

Alternate representations of the 2D continuous-domain photograph are the 2D surface representation and the contour reprentation shown in Figure (2.2).

If the value of a signal is assigned in proportion to some physically-measurable quantity (such as the gray-level intensity of light, as in this particular example), then the signal is said to be an **analog signal**. Therefore, a photograph is an *analog* continuous-domain 2D signal $x(t_1, t_2)$.

On The Physical Interpretation of the Continuous-Domain Assumption

From a physical point of view, the continuous-domain spatial property is an assumption that can only be made in the above example because the HVS *perceives* the intensity to be defined on a continuum of values in $\mathbf{R}^2_{\mathbf{rec}}$. We recognize that, in reality, this perception breaks down if we view the image at a sufficiently fine scale (under a microscope, for example). Then, we might observe that a gray tone is evidently made up of a collection of non-touching black circular dots on a white background in such a way that the level of the gray-tone is represented by the average density of the black dots, as illustrated in Figure (2.3). The HVS performs the 2D spatial averaging that allows these collection of circular dots to be perceived, from a sufficient distance, as a continuous-domain image.

There are many other examples where natural biological systems, as well as artificial cognitive systems, are unaware of fine spatial details, thereby providing the illusion, or perception, that the domain is continuous. The typical cathode ray tube (CRT) device, as used in television, is an example.

FIGURE 2.2 Representations of a Photograph $x(t_1, t_2)$

Doesn't fit (working on it).

FIGURE 2.3 Microscopic View of a Gray Tone in a Photgraphic Image

(see Fig. 2.2)

A 2D Mixed Domain Image: A Scanned Photograph

Transducers are used to convert photographic images to electronic form. For example, a common technique is to employ a scanning device to scan the photograph along closely-spaced horizontal rows, producing the so-called raster-scanned image as shown in Figure (2.4). Television cameras work in this way. The corresponding scanned image may be represented as a 2D mixed-domain spatial signal $x(t_1, n_2T)$, where

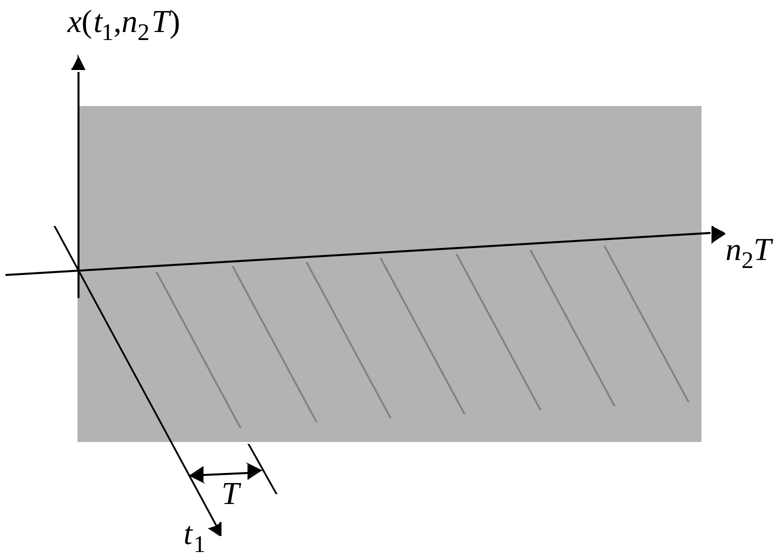
$$t_1 \in [0;T_{row}] \ , \ T_{row} \in \mathbf{R^1} \tag{2.4}$$

and

$$n_2 \in [0;(N_2 - 1)] \ , \ N_2 \in \mathbf{N^1} \ , \ T \in \mathbf{R^1} \tag{2.5}$$

The length of each row is T_{row} , the number of rows is N_2 , and the distance between each row is T . Clearly, this signal is continuous-domain in the row dimension t_1 and discrete-domain in the column dimension n_2 . It is therefore a *mixed-domain* signal and $dmn[x(t_1, n_2T)] \in \mathbf{R^1xN^1}$.

FIGURE 2.4 A Raster Scan Mixed-Domain 2D Signal $x(t_1, n_2T)$ Representing A Scanned Photograph



A 2D Analog Discrete-Domain Signal: The Rectangularly-Sampled Photograph

The above raster-scan image may be converted to a rectangularly-sampled version of the photograph by sampling each raster-scanned row at uniform spatial intervals, spaced distance T apart, resulting in the 2D analog discrete-domain signal representation $x(n_1T, n_2T)$. If we have N_1 samples per row and N_2 rows, then

$$dmn[x(n_1T, n_2T)] = ((n_1T)(n_2T))| \quad n_1 = [0;N_1], \; n_2 = [0;N_2]$$

We shall often normalize the distance T to unity and write the uniformly-sampled distance-normalized 2D discrete-domain signal as $x(n_1, n_2)$, in which case $dmn[x(n_1, n_2)] \in \mathbf{N}^2_{\mathbf{rec}}$ where $\mathbf{N}^2_{\mathbf{rec}} \equiv [((0)(0));((N_1)(N_2))]$ is a rectangular array of integer 2-tuples.

The complete sequence $\{x(n_1, n_2)\}$ is often represented as a matrix in which the element in row n_1 and column n_2 has value $x(n_1, n_2)$. Thus, the sequence is represented as the matrix

$$\begin{bmatrix} x(0,0) & x(0,1) & \dots & x(0,(N_2-1)) \\ x(1,0) & x(1,1) & \dots & x(1,(N_2-1)) \\ x(2,0) & x(2,1) & \dots & x(2,(N_2-1)) \\ \dots & \dots & \dots & \dots \\ x((N_1-1),0) & x(0,1) & \dots & x((N_1-1),(N_2-1)) \end{bmatrix} \tag{2.6}$$

The values of the elements of this matrix belong to some interval in $\mathbf{R}^{\mathbf{1}}$. *In this representation, the photograph is now nothing but a set of real numbers*. One might ask how we can reverse the process and view a photograph that is represented by the numbers in equatioon (2.6).

Viewing the 2D Discrete-Domain Analog Signal using Pixels

The matrix in equation (2.6) can easily be converted to a photograph, having domain $\mathbf{R}^{\mathbf{2}}_{\mathbf{rec}}$. Each element $x(n_1, n_2)$ of the matrix may be converted to a small uniform-intensity square tile having its centre at $((n_1 T)(n_2 T))$ in $\mathbf{R}^{\mathbf{2}}$. The light intensity of the gray-tone is proportional to the value o f $x(n_1, n_2)$.

The ROS of the resulting photograph is the whole of $\mathbf{R}^{\mathbf{2}}_{\mathbf{rec}}$ if the edges of each tile are of length T so that the tiles are just-touching tiles, as shown in Figure (2.5).

We may represent each matrix element $x(n_1, n_2)$ as the real-domain 2D impulse function of strength $x(n_1, n_2)$ given by

$$x(n_1, n_2)\delta(n_1 T, n_2 T) \in \mathbf{R}^{\mathbf{2}}_{\mathbf{rec}} \tag{2.7}$$

The resulting elemental tile, referred to as a *pixel* $p(t_1, t_2)$ may be described mathematically in terms of the product of two strips of width T as follows:

$$p(t_1, t_2) = [x(t_1, t_2)][u((t_1 - 0.5)T, t_2) - u((t_1 + 0.5)T, t_2 \qquad \text{intensity} \propto x(n_1, n_2)$$

$$[u(t_1, (t_2 - 0.5)T) - u(t_1, (t_2 + 0.5))]$$

The operation of converting the 2D real-domain impulse in equation (2.7) to the square real-domain pixel of equation (2.8) is the so-called **2D sample-hold** operation and is illustrated in Figure (2.5). The resulting photograph is shown in Figure (2.6).

It is sometimes convenient to display such a photograph as a 2D surface-histogram, as shown in Figure (2.6), where each bin represents a sample in the 2D sequence having height equal to the value of the sample.

FIGURE 2.5 The 2D Sample-Hold Operation That Converts A 2D Impulse To A 2D Pixel

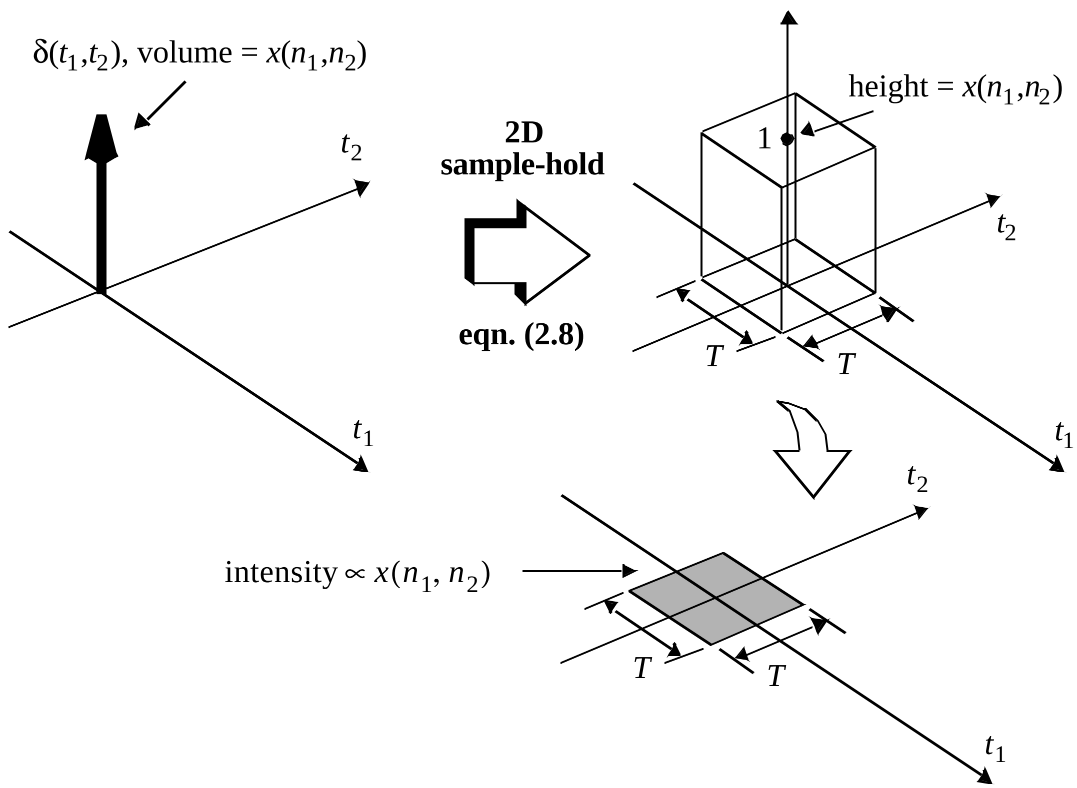
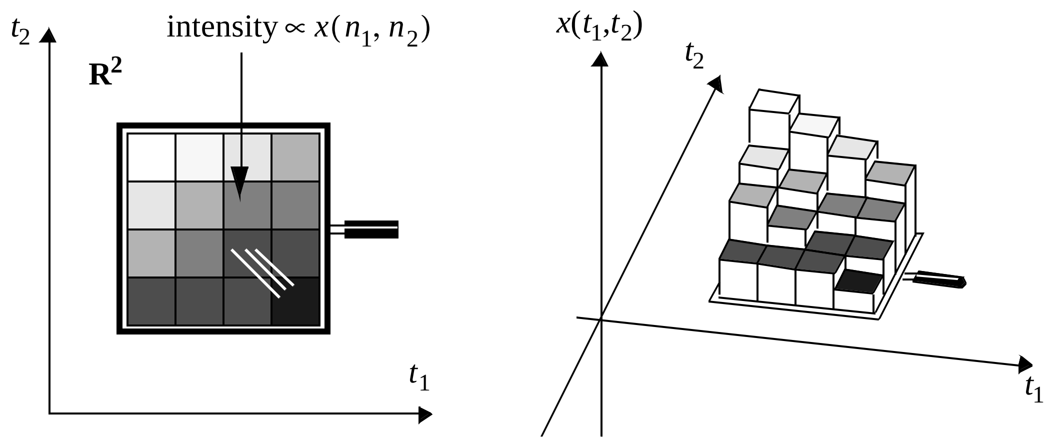


FIGURE 2.6 Two Representations of a Matrix of Numbers as a Photograph in $\mathbf{R}^2_{\mathbf{rec}}$



Spatial Perception

Spatial perception of natural scenes by the HVS is a very complex subject. In general terms, the HVS system evidently processes the details in a scene by recognizing the general shape of the principal objects in the scene, according to their general outlines and according to such attributes as their shading, texture, color and so on. This classification-approach requires that *the boundaries of objects and their interior attributes be processed as a matter of temporal priority*. Many computer vision systems emulate this approach. The realistic spatial perception of objects and the proper visual tracking of moving objects require that the *objects be large enough, in terms of the number of samples used to represent them, that they form recognizable shapes with recognizable shading and texture*. This rather obvious statement explains why *our* vision system is not so foolish as to attempt to process a scene by emulating the raster scan approach that is used in video cameras.

The recognition that we require large numbers of samples to represent outlines, texture and shading has significant practical implications. For example, on a typical television screen we might expect that many quite small recognizable objects will be composed of hundreds of samples. Thus, a square object consisting of 100 samples occupies less than $1cm^2$ in a square screen of size $(50)x(50)$ cm^2.having $(1000)x(1000)$ samples. Objects consisting of less than 25 samples will generally be difficult to recognize and will have little or no outline details and outlines will not be smooth, in general. For the interior texture or shading of an object to be recognized, many hundreds of samples are required.

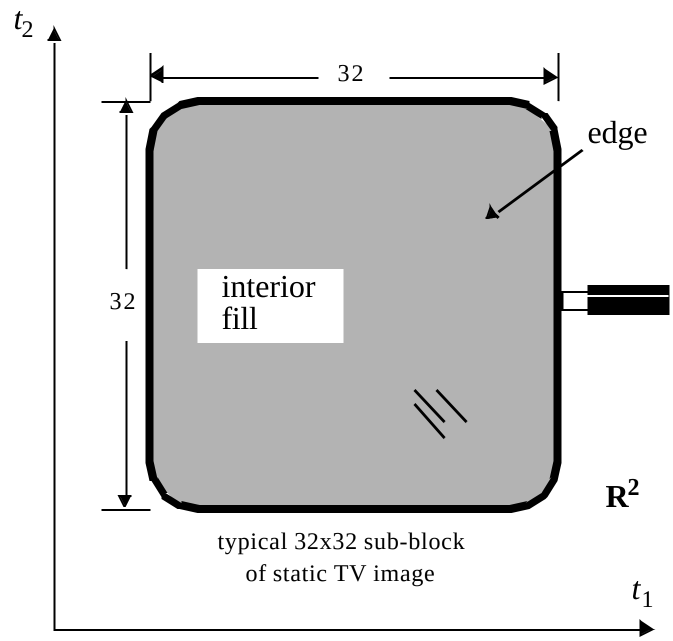
The above discussion implies that objects are generally supported by at least hundreds of samples although their textures may be represented by important detail at spatial distances down to a few samples. Shading requires hundreds of samples for realistic representation.

It is often a good approximation to assume that television images are composed of a collection of objects having mostly smooth outlines and interiors that have such attributes as shading, texture, brightness, etc.

Properties in Single-Frame Spatial Square Sub-blocks

If one divides a typical television frame into spatial square sub-blocks of a sufficiently small size, typically 32x32 samples, then most of these square sub-blocks cover two principal types of sub-images. They will cover only the interior of objects or they will cover short approximately-linear segments of the edges of objects, as shown in Figure (2.10). This suggests that two types of sub-images, or component images, are especially important. First, slowly varying intensities that represent shaded interiors of objects and, second, objects that represent segments of the edges of objects.

FIGURE 2.7 Spatial Sub-Blocks of a Typical Television Image



The Quantization and Coding of Image Values for Representation by a Machine

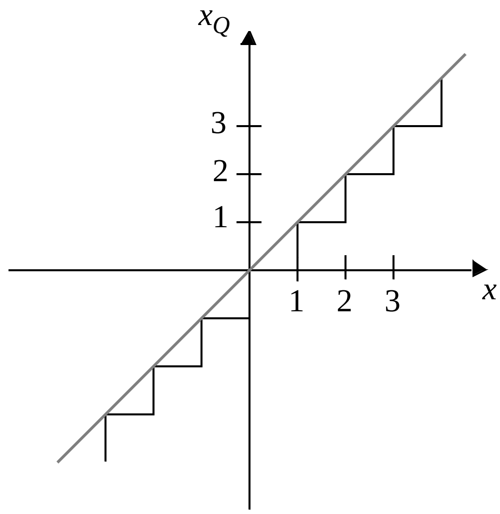
The above 2D discrete-domain analog version $x(n_1, n_2)$ of a photograph cannot generally be represented in a machine, such as a computer, because $x(n_1, n_2)$ may have any one of an *infinite number* of values on say the real interval $[0;1]$, representing the gray-scale from black to white. A machine, such as a computer, can only store a *finite number* of values, thereby forcing us to quantize the value $x(n_1, n_2)$ to some approximate value $x_Q(n_1, n_2)$. We represent the operation of quantization by means of the operator Q[.] so that

$$x_Q(n_1, n_2) = \mathrm{Q}[x(n_1, n_2)] \tag{2.9}$$

For example, the so-called negative-truncation operator Q[.] is shown in Figure (2.8). In this case, the operator Q[.] quantizes the value of the signal to the nearest entire integer that is less than or equal to the value $x(n_1, n_2)$, which is often written as

$$x_Q(n_1, n_2) = \lfloor x(n_2, n_2) \rfloor \tag{2.10}$$

FIGURE 2.8 Negative-Truncation Operator Q[.](2.8)



Some appropriate code must be used in the machine to represent each quantized value $x_Q(n_1, n_2)$. The quantized values, which are integers in this example, are generally coded according to a suitable number representation. Most commonly, the **fixed-point binary number code** or the **floating point binary number code** are employed. The binary number system is reviewed in Appendix ??. Binary methods convert the quantized signal values $\lfloor x(n_2, n_2) \rfloor$ to binary words, which are strings of 1-digits and 0-digits.

The fixed-point representation of non-negative numbers $\lfloor x(n_2, n_2) \rfloor$ is given by the values of the binary digits $b_i = [0;1] \in \mathbf{Z}^{\mathbf{1}}$ that satisfy the equation

$$(\lfloor x(n_2, n_2) \rfloor)_2 = \sum_{j=0}^{W-1} b_j(2^j) \tag{2.11}$$

where W is the wordlength of the binary representation (or word) and the subscript 2 on the left side implies the binary representation.

For example, if $x(n_1, n_2) = 14.91$ and $W = 8$ bits, then $\lfloor x(n_2, n_2) \rfloor = 14$ and, according to equation (2.11),

$$(\lfloor x(n_2, n_2) \rfloor)_2 = \{b_7 b_6 b_5 b_4 b_3 b_2 b_1 b_0\} = \{00001110\} \tag{2.12}$$

The digitized binary version of the image is therefore a matrix of 8 bit binary words, where each element in the matrix (2.6) is replaced with the binary word corresponding to the value of the element.

An image having numbers of row and column samples given by $N_1 = N_2 = 1024$ requires a total of $N_1 N_2 W$ bits for its binary representation using W-bit words. In this example, this corresponds to $(1024)(1024)(8) = 8388608$ bits. Clearly, this is a significant memory storage requirement, especially if we want to store large numbers of digitized images. For this reason, methods of compressing such memory storage requirements are very important and will be considered later in this book.

A Digitized Television Image

A digitized television image is simply a **temporal sequence** of the above 2D digitzed images and may be written as the quantized 3D signal

$$x(\mathbf{n}) = x(n_1, n_2, n_3) \tag{2.13}$$

where

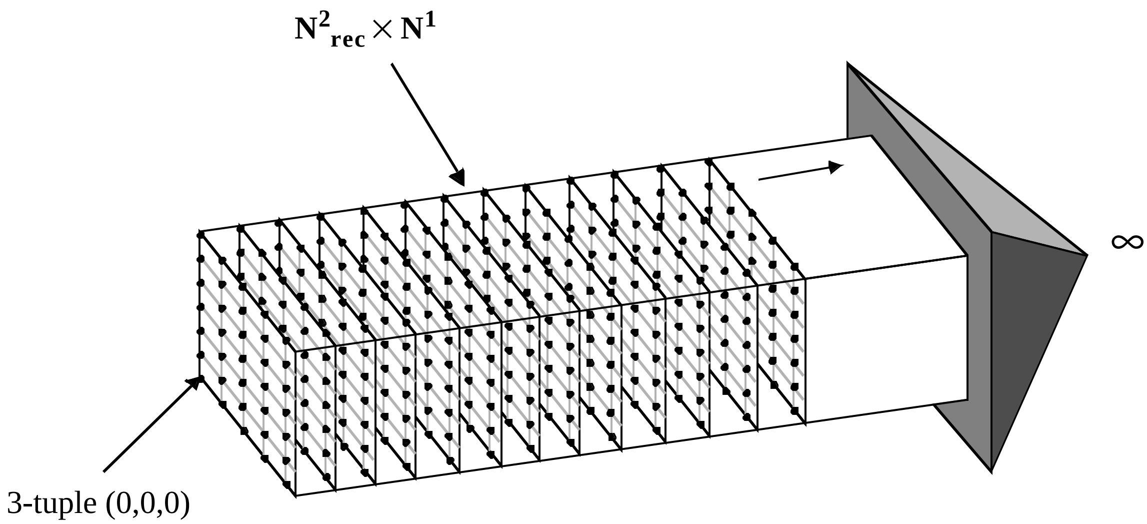
$$(n_1, n_2) \in \mathbf{N_{rec}} \text{ and } n_3 \in \mathbf{N^1} \tag{2.14}$$

The domain of a digitized television signal is of finite extent in the two spatial domain variables. However, in the temporal dimension n_3, it is (for all practical purposes) of infinite extent. The individual spatial images are referred to as frames and so the temporal index n_3 points to the n_3 th frame in the virtually-infinite sequence of frames. We may write

$$dmn[x(\mathbf{t})] = \mathbf{N_{rec}} \mathbf{x} \mathbf{N^1} \tag{2.15}$$

This discrete-domain is contained in a 3D solid rectangular extrusion of infinite length in the temporal direction, as shown in Figure (2.9) .

FIGURE 2.9 The 3D Domain of a Digitized Television Signal $x(\mathbf{t})$



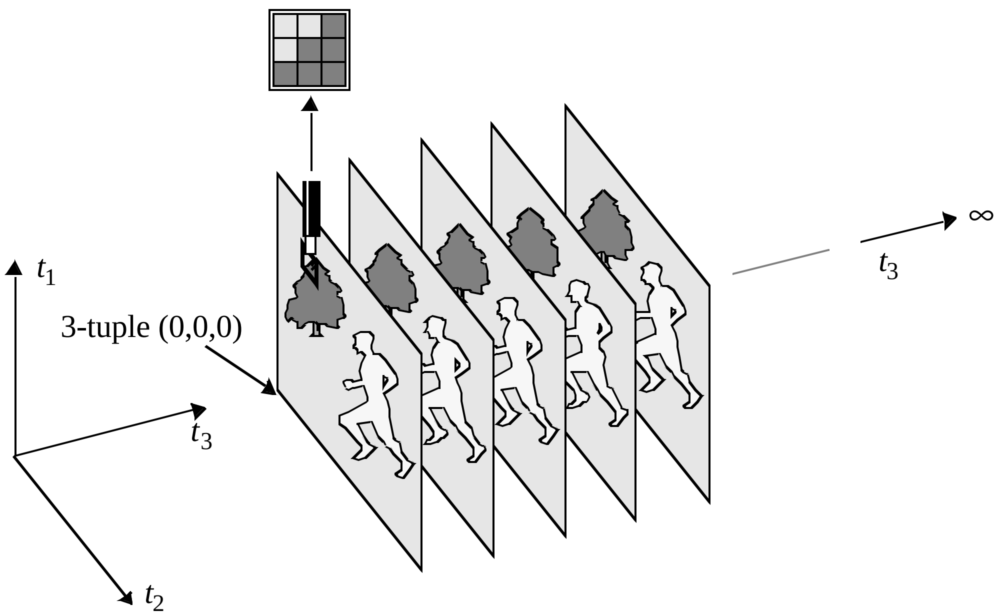
The range of $x(\mathbf{t})$ depends on the wordlength. For example, in the above example where the wordlength $W = 8$ bits,, the values of $x(\mathbf{t})$ lie on the closed integer interval [0 255] and so $(rng[x(\mathbf{t})] = [0;255]) \in \mathbf{N}^1$. In a practical situation, we might need to consider how to code a quantized signal that might lie outside this range.

We may sketch a digitized television image as shown in Figure (2.10) . The discrete nature of the temporal dimension is emphasized in this sketch, where individual frames are evident and drawn separately. The spatial samples are rectangularly-tiled because this is approximately the way we see them at close range.

Smooth Temporal Motion

It is very important to recognize that the HVS (and the design of the television system) require that the television image is, for the most part, changing sufficiently slowly with time that most parts of most consecutive frames, $x(t_1, t_2, t_3)$ and $x(t_1, t_2, t_3 - 1)$, differ only slightly from each other so that the sequence of images *retains the perception of natural-looking smooth motion*. This is another example where perception is different than reality in the sense that the temporal domain is perceived to be continuous, partly because the temporal averaging of the HVS often creates the illusion of smooth motion if the frame rate exceeds about 60 frames per second. The opticas of both the typical television camera and the typical CRT screen introduce further temporal smoothing.

FIGURE 2.10 Representation of a Digitized Television Image $x(n_1, n_2, n_3)$

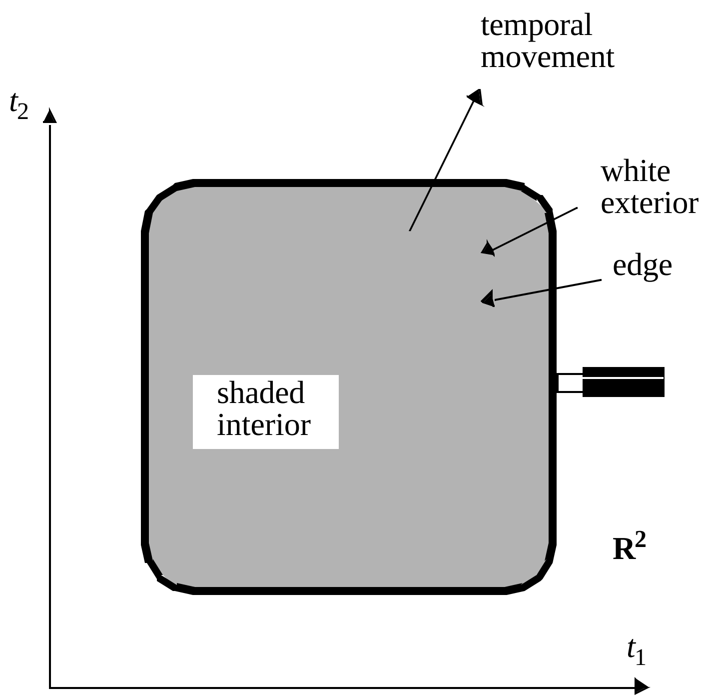


2.4 CONTINUOUS-DOMAIN 3D IMAGE OBJECTS

Many images are composed of complicated collections of objects, having characteristic outlines and characteristic interiors. In order to provide a framework for the mathematical representation of 3D images, it is useful to consider some elementary image objects having idealized properties. These objects may be employed as the components from which more realistic and complicated images are constructed.

For example, we have noted that single-frame sub-images might cover part of the interior of an object and exhibit a smoothly-varying spatial variation of intensity, corresponding to a shadowed region. Alternatively, a single-frame sub-image might cover an almost-linear segment of the curved edge of an object, in which case we might expect the intensity to change abruptly in directions normal to that edge. In both cases, these sub-images might be changing their spatial locations from frame to frame, as they move with time, as indicated in Figure (2.11).

FIGURE 2.11 Sub-Image Coverage of A Smooth Interior and of an Edge Segment



These observations motivate the following introduction of idealized mathematical models for simple 3D objects. We shall emphasize the 3D case, although the 2D case is easily considered by dropping the third dimesnion n_3 and the 4D case by obvious extension to the fourth dimension n_4 .

We shall consider here the continuous-domain and often pretend that the objects have support throughout the entire 3D region $\mathbf{R}^3$. In practise, it must be recognized that actual signals will have a more restricted domain, such as (2.15) above. Further, the signals will be sampled. The effects of finite extent and of sampling are important and must later be taken into account. For now, we ignore these effects.

2.4.1 A 3D Gate Pulse Signal of Infinite Extent in $\mathbf{R}^3$.

Consider the signal

$$x(\mathbf{t}) \equiv \begin{matrix} 1, (\forall \mathbf{t} \in \mathbf{R}^{\mathbf{3}}_{\mathbf{pulse}}) \\ 0, \mathit{otherwise} \end{matrix} \tag{2.16}$$

where the region of support $\mathbf{R}^{\mathbf{3}}_{\mathbf{pulse}}$ is defined as

$$\mathbf{R}^{\mathbf{3}}_{\mathbf{pulse}} \subset \mathbf{R}^{\mathbf{3}}\Big|, l_1 \le d_1 t_1 + d_2 t_2 + d_3 t_3 \le l_2, (l_1, l_2, d_1, d_2, d_3 \in \mathbf{R}^{\mathbf{1}}), (l_1 < l_2) \tag{2.17}$$

Recall that the equation

$$d_1 t_1 + d_2 t_2 + d_3 t_3 = l \tag{2.18}$$

describes a plane in $\mathbf{R}^{\mathbf{3}}$ having normals $\mathbf{d}$ that are given by the vectors

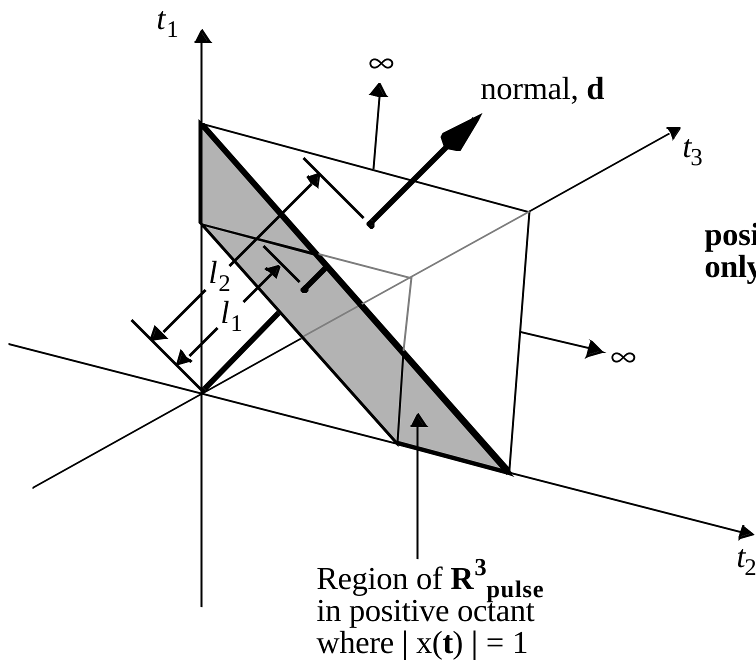
$$\mathbf{d} = \pm \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} \tag{2.19}$$

and having a perpendicular distance l from the origin, as shown in Figure (2.12). Therefore, the region $l_1 \le d_1 t_1 + d_2 t_2 + d_3 t_3 \le l_2$, as defined in equation (2.17), is that contained between the two 2D planes $d_1 t_1 + d_2 t_2 + d_3 t_3 = l_1$ and $d_1 t_1 + d_2 t_2 + d_3 t_3 = l_2$. This is a region of support $\mathbf{R}^{\mathbf{3}}_{\mathbf{pulse}}$ having the shape of a **3D slice** of uniform thickness , within which the defined signal $x(t_1, t_2, t_3)$ has the value 1 , as shown in Figure (2.12).

The signal is a 3D pulse of duration $(l_2 - l_1)$ in the direction defined by $\mathbf{d}$. It represents the type of signal that is ideally reflected from an infinitely distant target and detected, as a function of time t_3 , by a 2D spatially-rectangular array over the variables t_1 and t_2 in such applications as the detection of reflected radar or sonar signals . The pulse is observed, as a function of time t_3 , to move across the 2D spatial image in variables t_1 and t_2 . We shall return to the subjects of dynamic 2D spatial images. This signal might also represent a line-edge of thickness $l_2 - l_1$ that is moving with constant temporal velocity across the spatial plane.

This signal is clearly discontinuous-valued and therefore non-analytic.

FIGURE 2.12 The 3D Plane and the Continuous-Domain 3D Gate Pulse Signal of Infinite Extent



2.4.2 Exponential and Sinusoidal 3D Signals

If the reader is not completely familiar with the 1D exponential series $exp[x]$ and its relationships with the sine and cosine series $\sin(x)$ and $\cos(x)$, it is recommended that Appendix A be reviewed.

In order to describe the 3D exponential series, we shall require the 3D complex frequency vector

$$\mathbf{s} = [s_1, s_2, s_3]^{\mathbf{T}} \in \mathbf{C}^{\mathbf{3}} \tag{2.20}$$

where $s_i = \sigma_i + j\omega_i$, $i = 1, 2, 3$. We refer to

$$\omega = [\omega_1, \omega_2, \omega_3]^{\mathbf{T}} \in \mathbf{R}^{\mathbf{3}} \tag{2.21}$$

as the 3D (real) frequency vector. Given

$$\mathbf{t} \equiv [t_1, t_2, t_3]^{\mathbf{T}} \in \mathbf{R}^{\mathbf{3}}, \tag{2.22}$$

the **inner product** of the two vectors $\mathbf{s}$ and $\mathbf{t}$ is defined as

$$\mathbf{s} \cdot \mathbf{t} \equiv \mathbf{s}^{\mathbf{T}}\mathbf{t} = s_1 t_1 + s_2 t_2 + s_3 t_3 \in \mathbf{C}^{\mathbf{1}} \tag{2.23}$$

The 3D Complex Exponential Signal

The 3D *complex-valued s-domain* exponential series is defined as

$$x(\mathbf{t}) \equiv exp[\mathbf{s}^{\mathbf{T}}\mathbf{t}] = e^{\mathbf{s}^{\mathbf{T}}\mathbf{t}} \in \mathbf{C}^{\mathbf{1}} \quad , \mathbf{t} \in \mathbf{R}^{\mathbf{3}}, \mathbf{s} \in \mathbf{C}^{\mathbf{3}} \tag{2.24}$$

With $s_i \equiv \sigma_i + j\omega_i$, we shall often be interested in this signal over the region where $\sigma_i = 0$, $i = 1, 2, 3$, corresponding to the so-called **3D phasor** exponential signal

$$x(\mathbf{t}) = exp[j\mathrm{w}^{\mathbf{T}}\mathbf{t}] = e^{j\mathrm{w}^{\mathbf{T}}\mathbf{t}} \in \mathbf{C}^{\mathbf{1}} \tag{2.25}$$

The 3D Sinusoidal and Cosinusoidal Signals

Taking the real and imaginary parts of equation (2.25) gives

$$Re[exp[j\mathrm{w}^{\mathbf{T}}\mathbf{t}]] = \cos(\mathrm{w}^{\mathbf{T}}\mathbf{t}) = \cos(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3) \in \mathbf{R}^{\mathbf{1}} \tag{2.26}$$

and

$$Im[exp[j\mathrm{w}^{\mathbf{T}}\mathbf{t}]] = \sin(\mathrm{w}^{\mathbf{T}}\mathbf{t}) = \sin(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3) \in \mathbf{R}^{\mathbf{1}} \tag{2.27}$$

The domain of these two sinusoidal 3D signals, in terms of the independent variable $\mathbf{t}$, is $\mathbf{R}^{\mathbf{3}}$. Such signals are important component signals of 3D images. They appear in the next chapter, where they form the basis of the frequency domain description of 3D signals.

The 3D Sinusoidal Signal in $\mathbf{R}_{\mathbf{rec}}\mathbf{x}\mathbf{R}^{\mathbf{1}}$

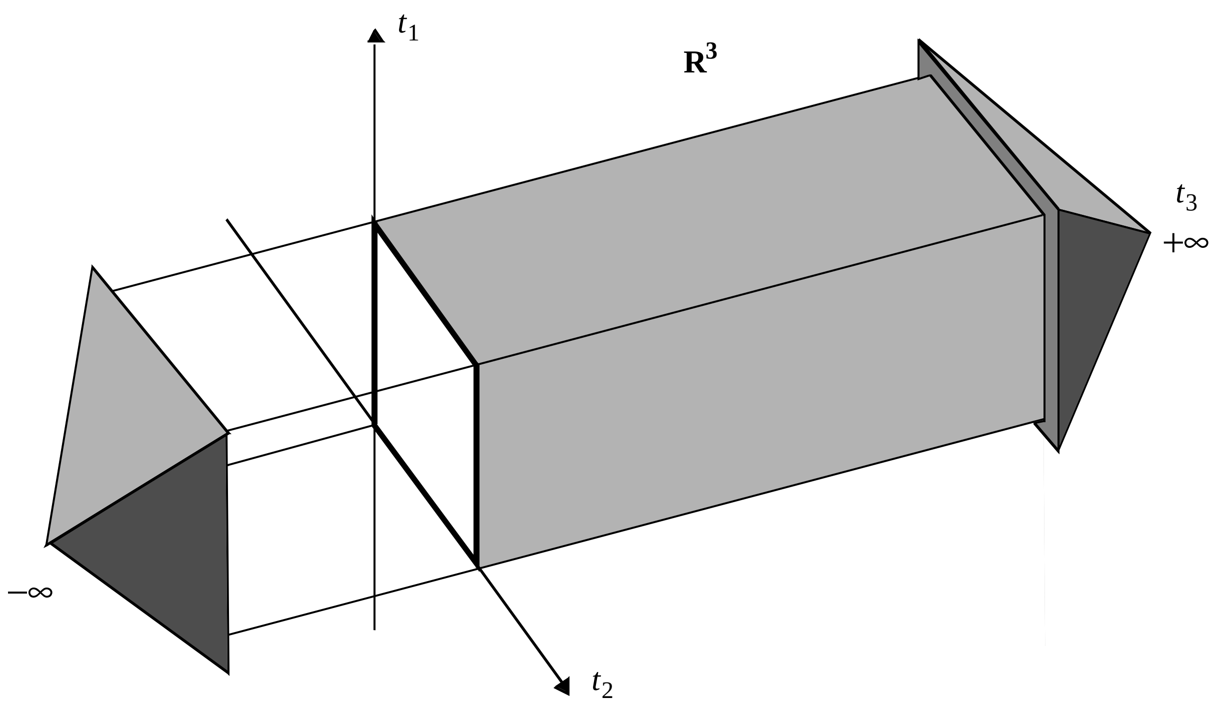
Consider the 3D signal

$$x(\mathbf{t}) \equiv \sin(\omega^{\mathbf{T}}\mathbf{t}) = \sin(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3), \mathbf{t} \in \mathbf{R_{rec}} \mathbf{x} \mathbf{R^1} \qquad (2.28)$$

as shown in Figure 2.13, where we interpret t_1 and t_2 as spatial variables and t_3 as the temporal variable.

This signal has a number of properties that are noteworthy. If viewed as a time-varying spatial image, the signal moves with constant temporal velocity across the spatial viewing area $\mathbf{R_{rec}}$. Throughout the 3D region $\mathbf{R_{rec}}\mathbf{x}\mathbf{R^1}$, its intensity is constant in the direction $\mathbf{d}$ and varies sinusoidally in the direction $\mathbf{p}$, and these two vectors are orthogonal. Signals that have these properties will be studied in further depth.

FIGURE 2.13 The 3D Sinusoidal Signal in $\mathbf{R_{rec}}\mathbf{x}\mathbf{R^1}$



In order to establish the mathematical framework for the discussion of various types of MD signals, we begin with a brief review of the MD geometry of lines and planes.

An MD signal that is duration-bounded in at least one dimension and duration-unbounded in at least one dimension is defined as *partially duration-bounded*. Such a signal could have a ROS that is of infinite extent in some of the dimensions and of finite extent in the remaining dimensions.

The analysis and design of signal processing systems is simplified by considering special classes of generic signals. These signals are widely used for such purposes as the generation of suitable test signals. Some of them occur as component signals in the output response signals of signal processing systems. The more important of these signals are now considered.

2.4.3 Step Functions

The MD continuous domain *fully causal unit step function* is defined as

(2.29)

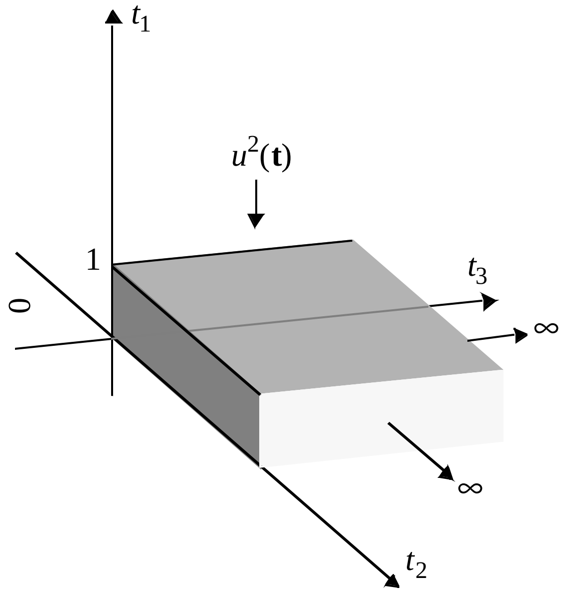
$$u^N(\mathbf{t}) \equiv \begin{matrix} 1, \mathbf{t} \in \mathbf{R}^{\mathbf{M}}_{\mathbf{causal}} \\ 0, otherwise \end{matrix} \tag{2.30}$$

where

$$\mathbf{R}^{\mathbf{M}}_{\mathbf{causal}} \equiv \bigcap_{k=1}^{N} t_k \geq 0 \tag{2.31}$$

That is, the region $\mathbf{R}^{\mathbf{N}}_{\mathbf{causal}}$ is the region of MD Euclidean space where the t_k are *all* non negative. This region is defined as the fully-causal region. The continuous domain 2D unit step function $u^2(\mathbf{t})$ is shown in Figure 2.10 . Such functions are useful for determining the transient behaviour of a system to the introduction of a step discontinuity and also as a multiplicative function for bounding the duration of a signal function $x(\mathbf{t})$; for example, $x(\mathbf{t})[u^M(\mathbf{t}) - u^M(\mathbf{t_0})]$ duration bounds $x(\mathbf{t})$ to some MD-rectangular region of support defined by the origin $\mathbf{t} = (0, 0, 0, \ldots, 0)$ and the far corner $\mathbf{t_0} = (t_{01}, t_{02}, t_{03}, \ldots, t_{0N})$.

FIGURE 2.14 The Continuous-domain 2D unit step function



2.4.4 Impulse and Knife Edge Functions

The MD continuous domain unit impulse function $\delta^N(\mathbf{t})$ is defined by the following three constraints:

$$\delta^N(\mathbf{t}) = 0, \forall \mathbf{t} | (\mathbf{t} \neq \mathbf{0}) \tag{2.32}$$

and

$$\int_{-\infty}^{\infty} \delta^N(\mathbf{t}) \mathbf{dt} = 1 \tag{2.33}$$

where the integral implies MD integration over the entire region $\mathbf{R^N}$. This function is shown in Figure 2.10 for the 2D case. Many other types of impulse functions may be defined for the MD case. For example, the 2D horizontal and vertical impulse slices, or 2D **knife edge functions**, shown in Figure 2.10 have infinite magnitudes along one of the axes.

Discrete domain versions of the impulse and knife edge functions may be defined. We shall be particularly interested in the distance normalized discrete domain MD unit impulse function $\delta^N(\mathbf{n})$ which is defined as follows

$$\delta^N(\mathbf{n}) \equiv \begin{matrix} 0, \forall \mathbf{n} | \mathbf{n} \neq \mathbf{0} \\ 1, \mathbf{n} = \mathbf{0} \end{matrix} \tag{2.34}$$

and is shown in Figure 2.10 for the 2D case.

2.5 MD Lines and Planes

The MD Line

We define the MD line (or **hyperline**) as the region $\mathbf{R}^{\mathbf{N}}_{\mathbf{line}} \subset \mathbf{R}^{\mathbf{N}}$ where

$$\mathbf{R}^{\mathbf{N}}_{\mathbf{line}} \equiv \mathbf{t} | = \frac{t_1 - \gamma_1}{\beta_1} = \frac{t_2 - \gamma_2}{\beta_2} = \frac{t_3 - \gamma_3}{\beta_3} = \dots = \frac{t_d - \gamma_d}{\beta_{1d}} = \dots$$

$$\dots = \frac{t_m - \gamma_m}{\beta_m}, (\beta_{1d}, \gamma_d, d \in \boldsymbol{R}^{\mathbf{1}}, 0 \leq d \leq 1) \tag{2.35}$$

The direction of this MD line in $\mathbf{R}^{\mathbf{2}}$ is given by the direction vector

$$ɔ = \left[\beta_1\ \beta_2\ \beta_3\ \dots\ \beta_d\ \dots\ \beta_N\right]^{\mathbf{T}} \tag{2.36}$$

and the MD line passes through the N-tuple

$$g = (\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_d, \dots, \gamma_N) \tag{2.37}$$

The **unit vectors** (that is, vectors of unit length) that point in the same direction as the line are given by

$$\mathbf{n} = \left[\frac{\beta_1}{\|b_1\|_2}\ \frac{\beta_2}{\|b_2\|_2}\ \frac{\beta_3}{\|b_3\|_2}\ \dots\ \frac{\beta_d}{\|b_d\|_2}\ \dots\ \frac{\beta_N}{\|b_N\|_2}\right]^{\mathbf{T}} \tag{2.38}$$

The operator $\|b\|_2$ on the vector β is referred to as the Euclidean Norm (or Euclidean length) of the vector and is therefore defined by the Pythagorean expression

$$\beta = \sqrt{\beta_1^2 = \beta_2^2 = \beta_3^2 = \ldots = \beta_d^2 = \ldots = \beta_N^2} \tag{2.39}$$

The 3D Straight Line

For $N = 3$, the straight line $\mathbf{R}^{\mathbf{3}}_{\mathbf{line}}$ in 3D Euclidean space is given by

$$\mathbf{R}^{\mathbf{3}}_{\mathbf{line}} \equiv \mathbf{t} \mid = \frac{t_1 - \gamma_1}{\beta_1} = \frac{t_2 - \gamma_2}{\beta_2} = \frac{t_3 - \gamma_3}{\beta_3} \tag{2.40}$$

This line passes through the 3-tuple $g = (\gamma_1, \gamma_2, \gamma_3)$.

The radian angles between the line $\mathbf{R}^{\mathbf{3}}_{\mathbf{line}}$ and the three axes are easily shown by simple geometry to be given by

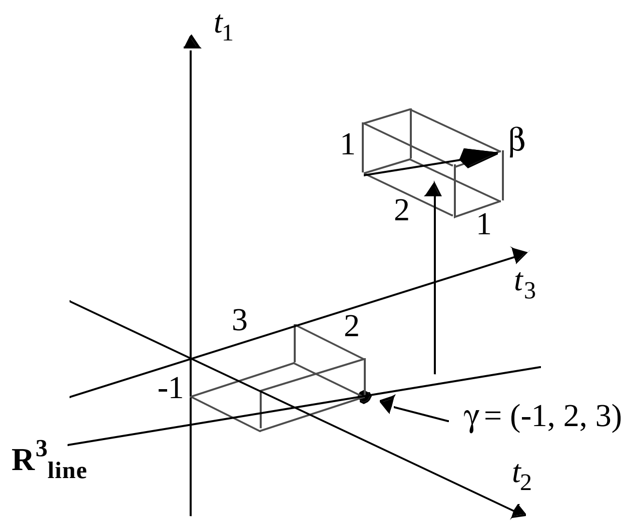
$$\theta_k = \operatorname{acos}\left(\frac{\beta_k}{\|b\|^2}\right) \tag{2.41}$$

where $k = 1, 2, 3$ and $b = \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}$ is the Euclidean Norm of the vector $b = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}^T$. The angles θ_k are defined as the **direction cosines** of the line because their cosines, according to the above equation, are equal to the three directional components of $\boldsymbol{n}$.

Consider a numerical example with $\beta_1 = 1,\ \beta_2 = 2,\ \beta_3 = 1$, $\gamma_1 = -1,\ \gamma_2 = 2,\ \gamma_3 = 3, \|\beta_2\| = \sqrt{6}$, as shown in Figure 2.15. We have $\beta = \sqrt{6}$, the line passes through the 3-tuple (-1 2 3) and the unit vector $\boldsymbol{n}$ in the direction of the line is $(1/\sqrt{6},\ 2/\sqrt{6},\ 1/\sqrt{6})$. The direction cosines are given by the 3-tuple of angles [1] $\theta = (\operatorname{acos}(1/\sqrt{6}),\ \operatorname{acos}(2/\sqrt{6}),\ \operatorname{acos}(1/\sqrt{6}))$

1.

FIGURE 2.15 A 3D straight line with $\beta_1 = 1,\ \beta_2 = 2,\ \beta_3 = 1$



The MD Plane

The concept of a MD plane, sometimes referred to as a hyperplane for $N > 3$, is a direct extension of the usual formulation of a 3D plane. We define the MD plane, or hyperplane, $\mathbf{R}^{\mathbf{M}}_{\mathbf{plane}}$ as the region

$$\mathbf{R}^{\mathbf{M}}_{\mathbf{plane}} \subset \mathbf{R}^{\mathbf{M}}\big|_{\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 \ldots + \alpha_d t_d \ldots + \alpha_N t_N = l} \tag{2.42}$$

where l and α_d, $1 \le d \le N$, are constants.

The **MD normal** to the above hyperplane is defined as the MD vector

$$\mathbf{n} = \pm\left[\frac{\alpha_1}{\|\alpha\|_2}\frac{\alpha_2}{\|\alpha\|_2}\frac{\alpha_3}{\|\alpha\|_2}\cdots\frac{\alpha_d}{\|\alpha\|_2}\cdots\frac{\alpha_m}{\|\alpha\|_2}\right]^{\mathrm{T}} \tag{2.43}$$

The 3D Plane

The **3D plane** is the region where

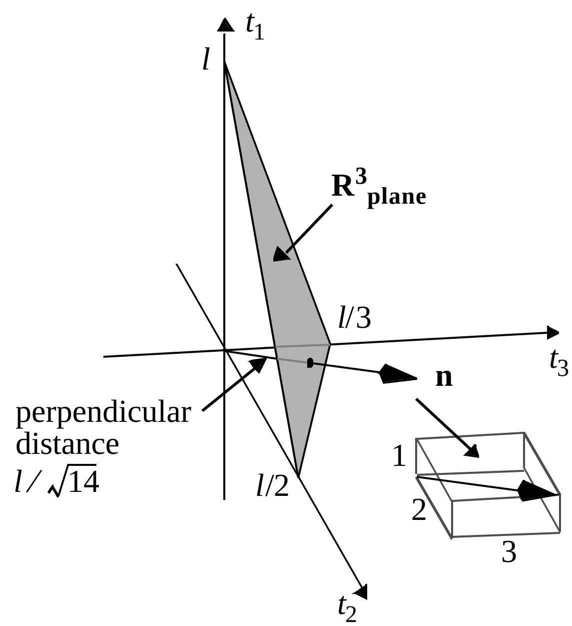
$$\mathbf{R}^3_{\mathbf{plane}} \subset \mathbf{R}^3\big|_{\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 ... + \alpha_d t_d ... + \alpha_N t_N = l} \tag{2.44}$$

It may be shown that the perpendicular distance between the plane and the origin is equal to l. The two 3D normals to the plane are given by

$$\mathbf{n} = \pm\left[\frac{\alpha_1}{\|\alpha\|_2} \frac{\alpha_2}{\|\alpha\|_2} \frac{\alpha_3}{\|\alpha\|_2}\right]^{\mathbf{T}} \tag{2.45}$$

Consider a numerical example with $\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3,\ l = 1$ as shown in Figure 2.16. The plane $\mathbf{R^3}_{\mathbf{plane}}$ is perpendicular distance 1 from the origin and the two unit normals to the plane are given by $\mathbf{n} = \pm\left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]$.

FIGURE 2.16 A 3D plane with $\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3,\ l = 1$



2.6 MD COORDINATE TRANSFORMATION

Let the independent MD N-element vector variable **t** be pre-multiplied by a real $N \times N$ matrix **A** to give the new N-element column vector **u** according to

$$\mathbf{u} = \mathbf{A}\mathbf{t}. \tag{2.46}$$

The vector **t** is subjected here to a linear transformation according to the transformation matrix **A** . It is transformed into the vector **u** . Clearly, the N row elements of **u** are linear combinations of the N row elements of **t** . **Such linear transformations are fundamentally important in signal processing**.

2.6.1 Rotational Transformation about One Axis

Consider the 2D coordinate vector $\mathbf{t} = \begin{bmatrix} t_1 & t_2 \end{bmatrix}^{\mathbf{T}}$ and a new rotated coordinate vector $\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^{\mathbf{T}}$, shown in Figure 2.17 , where the latter is obtained from the former by a clockwise rotation of θ_1 radians. Then it is easily shown that

$$\mathbf{u} = \mathbf{R}_1\mathbf{t} \tag{2.47}$$

where

$$\mathbf{R_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{2.48}$$

and is referred to as the rotation matrix for a 1D system. This rotation has the effect of rotating a 2D coordinate system about the first axis t_1 by θ_1 radians.

Similarly, in the case of a MD coordinate system, a coordinate rotation $\mathbf{R_d}$ about the d^{th} axis by θ_d is achieved by means of the transformation matrix

$$\mathbf{R_d} = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . & . & . & . & 0 & 0 \\ 0 & 1 & 0 & 0 & . & . & . & . & . & . & . & 0 \\ 0 & 0 & 1 & 0 & 0 & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 1 & 0 & 0 & . & . & . & . & . & . \\ . & . & 0 & 0 & 1 & 0 & 0 & 0 & . & . & . & . \\ . & . & . & 0 & 0 & \cos\theta_d & -\sin\theta_d & 0 & . & . & . & . \\ . & . & . & . & 0 & \sin\theta_d & \cos\theta_d & 0 & 0 & . & . & . \\ . & . & . & . & 0 & 0. & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & 0 & 0 & 1 & 0 & . & . \\ . & . & . & . & . & . & . & 0 & 0 & 1 & 0 & 0 \\ 0 & . & . & . & . & . & . & . & 0 & 0 & 1 & 0 \\ 0 & 0 & . & . & . & . & . & & . & 0 & 0 & 1 \end{bmatrix} \tag{2.49}$$

2.6.2 MD Rotational Transformations about Multiple Axes

Let us apply $N-1$ successive transformations $\mathbf{R_1}$, then $\mathbf{R_2}$, then $\mathbf{R_3}$, ... then $\mathbf{R}_{N-1}$ to sequentially rotate the coordinate system about $N-1$ of the axes, one at a time; first about the axis t_1, then about the axis t_2 and so on until the rotation has been performed about $N-1$ of the axes. This general rotation of the MD coordinate system is therefore given by the $N \times N$ rotational matrix $\mathbf{R}$ where

$$R_{N-1}R_{N-2}\ldots R_2R_1 \tag{2.50}$$

and

$$\mathbf{U} = R_{N-1}R_{N-2}\ldots R_2R_1\mathbf{t} = \mathbf{R}\mathbf{t} \tag{2.51}$$

The single axis $N \times N$ rotational transformation matrix $\mathbf{R_d}$ has the property that it is skew symmetric and its determinant $\det[\mathbf{R_d}] = 1$. It follows that

$$\det[\mathbf{R}] = \prod_{d=1}^{d=N-1} \det[\mathbf{R_d}] = 1 \tag{2.52}$$

Example 9

Rotations about the first and second axes t_1 and t_2 correspond to

$$\mathbf{R}_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix} \tag{2.53}$$

and therefore the general 3D rotation can be written as

$$\mathbf{R} = \mathbf{R}_1\mathbf{R}_2 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \cos\theta_2\sin\theta_1 & \cos\theta_2\cos\theta_1 & -\sin\theta_2 \\ \sin\theta_2\sin\theta_1 & \sin\theta_2\cos\theta_1 & \cos\theta_2 \end{bmatrix} \tag{2.54}$$

corresponding to the 3D rotational geometry shown in Figure 2.18.

2.6.3 Orthogonal Transformations

This section provides a review of some properties of vectors and matrices that are especially relevant to MD coordinate transformations and also to later considerations regarding the orthogonality of signals. The material is not essential for an understanding of this chapter and therefore the reader may choose to omit this section at this time.

The vector components t_d in equation ( ) are usually thought of, at least in the 3D case, as forming mutually perpendicular axes. Vector components that are mutually perpendicular are said to be **orthogonal**. So far, we have perhaps intuitively understood that any *N*-tuple location in $\mathbf{R}^\mathbf{N}$ can be found by specifying all *N* components t_d. We now formalize and generalize some of these concepts.

Real Orthogonal Vectors

Consider the *real* vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots, \mathbf{v_N}$, and the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_d, \ldots, \alpha_N$. Now, if

$$\alpha_1 v_\mathbf{1} + \alpha_2 v_\mathbf{2} + \alpha_3 v_\mathbf{3} + \ldots + \alpha_j v_\mathbf{j} + \ldots + \alpha_N v_\mathbf{N} \neq 0 \tag{2.55}$$

unless $\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_d = \ldots = \alpha_N = 0$, then the vectors $\alpha_1, \alpha_2, \ldots, \alpha_d, \ldots, \alpha_N$ are **linearly independent**.

If every MD vector **w** in $\mathbf{R}^\mathbf{N}$ can be expressed in terms of some linear combination

$$\mathbf{w} = \alpha_1 v_{\mathbf{1}} + \alpha_2 v_{\mathbf{2}} + \alpha_3 v_{\mathbf{3}} + \ldots + \alpha_j v_{\mathbf{j}} + \ldots + \alpha_N v_{\mathbf{N}} \tag{2.56}$$

by making a suitable choice for the N real weights α_j, then the set of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots, \mathbf{v_N}$ is said to **span the space $\mathbf{R^N}$**.

If the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots, \mathbf{v_N}$ are linearly independent and if they span the space $\mathbf{R^N}$, then they are said to form a **basis** for the space $\mathbf{R^N}$. It is easily shown that the weights $\alpha_1, \alpha_2, \ldots, \alpha_d, \ldots, \alpha_N$ that satisfy equation (2.56) are unique; that is, no other combination of the vectors equal $\mathbf{w}$. The number of basis vectors, in this case N, is defined as the **dimension** of the space of $\mathbf{w}$.

The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as $\mathbf{u^T v}$. The two vectors $\mathbf{u}$ and $\mathbf{v}$ are defined to be **orthogonal** if their inner product is zero; that is, if

$$\mathbf{u^T v} = 0 \tag{2.57}$$

We define a set of m vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots, \mathbf{v_N}$ to be a **mutually orthogonal** set if each of the vectors $\mathbf{v_j}$ is orthogonal with the other $(N\text{-}1)$ vectors, $\mathbf{v}_i$, $i \neq j$. Therefore, N mutually orthogonal vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots, \mathbf{v_N}$ have the property that

$$\mathbf{v_i^T v_j} = 0, \quad \mathbf{i \neq j} \tag{2.58}$$

$$\mathbf{v_i^T v_j} = \|\mathbf{v_i}\|_2 \quad , \quad \mathbf{i \neq j} \tag{2.59}$$

We may think of each one of these vectors as pointing in the direction of one of the m mutually perpendicular coordinate axes in $\mathbf{R^N}$. The above mutually orthogonal vectors may each be scaled to vectors of unit length (that is, unit vectors) by dividing through by $\|\mathbf{v_i}\|_2$, in which case they are defined as an **orthonormal** set and satisfy the **orthonormal equations**

$$\mathbf{v_i^T v_j} = 0, \quad \mathbf{i \neq j} \tag{2.60}$$

$$\mathbf{v_i^T v_j} = 1, \quad \mathbf{i = j} \tag{2.61}$$

A basis of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots, \mathbf{v_N}$ that satisfies the above equations is an **orthonormal basis**. The most important orthonormal basis is the so-called **standard basis**, or **unit basis**, which is given by the following N vectors

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ .. \\ .. \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ .. \\ .. \\ 0 \\ 0 \end{bmatrix}, \ldots, \mathbf{e_N} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ .. \\ .. \\ 0 \\ 1 \end{bmatrix} \tag{2.62}$$

each of unit length and each pointing in the direction of one coordinate axis.

Orthogonal Matrices: Let **Q** be the real $N \times N$ matrix

$$\mathbf{Q} = [\mathbf{v_1}\ \mathbf{v_2} \ldots \mathbf{v_j} \ldots \mathbf{v_N}] \tag{2.63}$$

where the columns $\mathbf{v_j}$ therefore have dimension $N \times 1$. We define an **orthogonal matrix** Q as a real square matrix having columns that are orthonormal. It is easily shown, using the orthonormal property of the columns in equation ( ), that

$$\mathbf{Q^T Q} = \mathbf{Q Q^T} = \mathbf{I} \tag{2.64}$$

where **I** is the $N \times N$ identity matrix. Also, since

$$\mathbf{Q^{-1} Q} = \mathbf{I} \tag{2.65}$$

it follows that

$$\mathbf{Q^T} = \mathbf{Q^{-1}} \tag{2.66}$$

and

$$\mathbf{Q^T} = \mathbf{Q} \tag{2.67}$$

Equation ( ) describes an important property of orthogonal matrices; that is, **the transpose of an orthogonal matrix is equal to its inverse**.

Determinant of an orthogonal matrix: It is easily shown that, for any two matrices **A** and **B**,

$$det(\mathbf{A})det(\mathbf{B}) = det(\mathbf{AB}) \tag{2.68}$$

It then follows from equations ( ), ( ), ( ) and ( ) that, for any orthogonal matrix **Q**,

$$\det(\mathbf{Q}^{-1})\det(\mathbf{Q}) = \det(\mathbf{Q}^{\mathbf{T}})\det(\mathbf{Q}) = 1 \tag{2.69}$$

from which it follows directly that

$$det(\mathbf{Q}) = \pm 1 \tag{2.70}$$

Rotational Property of Orthogonal Matrices: We are now in a position to derive an especially useful property of orthogonal transformations. Consider that two vectors, $\mathbf{x_1}$ and $\mathbf{x_2}$, are both subjected to the same orthogonal transformation $\mathbf{Q}$; that is

$$\mathbf{y_1} = \mathbf{Qx_1} \text{ and } \mathbf{y_2} = \mathbf{Qx_2} \tag{2.71}$$

We want to find out the effect of the orthogonal transformation $\mathbf{Q}$ on the inner product; that is, to compare $\mathbf{x_1^T x_2}$ with $\mathbf{y_1^T y_2}$. Using the elementary property of transposes that $(\mathbf{ab})^{\mathbf{T}} = \mathbf{b^T a^T}$, we write

$$\mathbf{y_2} = (\mathbf{Qx_1})^{\mathbf{T}}(\mathbf{Qx_2}) = (\mathbf{x_1^T Q^T})(\mathbf{Qx_2}) = \mathbf{x_1^T}(\mathbf{Q^T Q}) \tag{2.72}$$

and, using equation ( ), this simplifies to $\mathbf{y_1^T y_1} = \mathbf{x_1^T x_1}$.

We have shown that **inner products of real vectors are invariant under orthogonal transformation**. A second result follows by choosing $\mathbf{x_1} = \mathbf{x_2}$ (so that $\mathbf{y_1} = \mathbf{y_2}$) and the above equation simplifies to

$$\mathbf{y_1^T y_1} = \mathbf{x_1^T x_1} \tag{2.73}$$

which, using the property of matrices that $\mathbf{a^T a} = \|\mathbf{a}\|_2$, further simplifies to

$$\|\mathbf{y_1}\|_2 = \|\mathbf{x_1}\|_2 \tag{2.74}$$

Therefore the **orthogonal transformation Q preserves vector lengths** (that is, Euclidean norms). The length of a vector in the coordinate system **t** is the same as its length in the new coordinate system $\mathbf{u} = \mathbf{Qt}$ for real square **Q** if and only if **Q** is orthogonal. This implies that, in 3D and 2D Euclidean spaces, the orthogonal transformation is a **rotation** of the vector about the origin.

This implies that the matrices $\mathbf{R_i}$ and $\mathbf{R_d}$ in Section ?.? are real orthogonal matrices. The reader may verify this by checking that the inner products of their column vectors satisfy equation ( ).

Unitary Complex Vectors and Unitary Complex Matrices: The above analysis changes only slightly when the vectors are allowed to be complex. For example, in the above definition of linear independence, the weights a_d and the vectors are allowed to be complex. The **inner product of two complex vectors x** and **y** is defined as $(\mathbf{x}^*)^{\mathbf{T}}\mathbf{y}$, where the superscript asterisk indicates complex conjugation. We will denote the operations of conjugation followed by transposition with a superscript dagger † so that we may write

$$\mathbf{x}^\dagger \equiv (\mathbf{x}^*)^{\mathbf{T}} \tag{2.75}$$

Two complex vectors are defined to be orthogonal if their inner product is zero; that is, if

$$(\mathbf{x})^\dagger \mathbf{y} = 0 \tag{2.76}$$

The length of a complex vector **x** is defined as the inner product of **x** with itself

$$\|\mathbf{x}\|_2 = (\mathbf{x}^\dagger \mathbf{x}) \tag{2.77}$$

Orthonormal, or unitary, complex vectors are defined in the same way as for real vectors. That is, a system of complex vectors $\mathbf{y_1}, \mathbf{y_2}, ¼, \mathbf{y_j}, ¼, \mathbf{y_N}$ is orthonormal , or **unitary**, if

$$\mathbf{x_i}^\dagger \mathbf{x_j} = 0, \quad \mathbf{i} \neq \mathbf{j} \tag{2.78}$$

$$\mathbf{x_i}^\dagger \mathbf{x_j} = 1, \quad \mathbf{i} = \mathbf{j} \tag{2.79}$$

If the columns of a square complex matrix **Q** are unitary, then the matrix is said to be a **unitary matrix**.

It is easily shown that a unitary matrix has the important property that

$$\mathbf{Q}^\dagger = \mathbf{Q}^{-1} \tag{2.80}$$

It is now easy to show that the transformation of a complex $N \times 1$ vector **x** by a complex unitary $N \times N$ matrix **Q** preserves lengths. Let

$$\mathbf{y} = \mathbf{Qx} \tag{2.81}$$

From equation ( ), the squared length of the transformed complex vector **y** is given by

$$(\|\mathbf{y}\|_2)^2 = \mathbf{y}^\dagger\mathbf{y} = (\mathbf{Q}\mathbf{x})^\dagger(\mathbf{Q}\mathbf{x}) = (\mathbf{x}^\dagger\mathbf{Q}^\dagger)(\mathbf{Q}\mathbf{x})$$

$$= \mathbf{x}^\dagger(\mathbf{Q}^\dagger\mathbf{Q})\mathbf{x} = \mathbf{x}^\dagger(\mathbf{Q}^{-1}\mathbf{Q})\mathbf{x} = \mathbf{x}^\dagger\mathbf{x} = \|\mathbf{x}\|_2 \tag{2.82}$$

so that $\|\mathbf{y}\|_2$ is equal to $\|\mathbf{x}\|_2$, proving that the length of a complex vector is preserved under a unitary transformation.

This completes the brief review of properties of matrices.

2.6.4 Linear Trajectory MD Signals

Linear Trajectory (LT) signals are an especially important class of MD signals. They occur in many kinds of signal processing systems, including television, radar, and seismic signal processing.

A MD signal $x(\mathbf{t})$ is a Linear Trajectory (LT) signal if there exists a constant MD vector n $\in R^{\mathbf{N}}$ such that the directional derivative [redacted] is zero everywhere in the domain of the signal.

This definition implies that a LT signal $x(\mathbf{t})$ is constant along all MD lines having the same direction as the MD vector **n** . In the 3D case, for example, a LT signal is constant along all *straight* lines having a particular direction **n**. We shall therefore refer to the unit vector in the direction of **n** as the **constant signal vector** . The direction of **n** is the **constant signal direction** of the LT signal. The signal $\sin(\omega_1 t_1 + \omega_2 t_2),\quad \mathbf{t} \in \mathbf{R}^2$, that was considered in Example 16, is an example of a 2D LT signal because it has constant value along all 2D lines having direction $arctan(-\omega_1/\omega_2)$ corresponding to the constant signal vector

$$\mathbf{n} = \left[\frac{\omega_2}{\|w\|_2}\;\frac{\omega_1}{\|w\|_2}\right]^{\mathbf{T}} \tag{2.83}$$