CHAPTER 3 Linear Trajectory and Plane Wave 3D Signals

3.1 LINEAR TRAJECTORY MD SIGNALS

Linear Trajectory (LT) signals are an especially important class of MD signals. They occur in many kinds of signal processing systems, including television, radar, and seismic signal processing.

A MD signal $x(\mathbf{t})$ is a Linear Trajectory (LT) signal if there exists a constant MD vector $\mathbf{n} \in \mathbf{R}^N$ such that the directional derivative $\partial x(\mathbf{t}) / (\partial \mathbf{n})$ is zero everywhere in the domain of the signal}.

This definition implies that a LT signal $x(\mathbf{t})$ is constant along all MD lines having the same direction as the MD vector $\mathbf{n}$. In the 3D case, for example, a LT signal is constant along all *straight* lines having a particular direction **n**. We shall therefore refer to the unit vector in the direction of **n** as the **constant signal vector** . The direction of **n** is the **constant signal direction** of the LT signal. The signal $\sin(\omega_1 t_1 + \omega_2 t_2)$, $\mathbf{t} \in \mathbf{R}^2$, that was considered in Example 16, is an example of a 2D LT signal because it has constant value along all 2D lines having direction $arctan(-\omega_1 / \omega_2)$ corresponding to the constant signal vector

$$\mathbf{n} = \begin{bmatrix} \dfrac{\omega_2}{\|\omega_2\|_2} & \dfrac{-\omega_1}{\|\omega_2\|_2} \end{bmatrix}^{\mathbf{T}} \tag{3.1}$$

3.1.1 The 3D Linear Trajectory Signal

Suppose that a 3D LT signal $x(\mathbf{t})$ has a direction of constant intensity given by $\mathbf{n}$ and that we consider any two parallel 3D planes $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane1}} \subset \mathbf{R}^{\mathbf{3}}$ and $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane2}} \subset \mathbf{R}^{\mathbf{3}}$ having the normal $\mathbf{n}$, as shown in Figure (3.1). Then it follows that the value of the signal $x(\mathbf{t})$ at its intersection with the plane $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane1}}$ is exactly the same as the value of $x(\mathbf{t})$ at its intersection with the plane $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane2}}$. 3D LT signals $x(\mathbf{t})$ can be considered as *propogating without variation* in the direction $\mathbf{n}$, as shown in Figure (3.1).

FIGURE 3.1 Intersections of A Linear Trajectory (LT) Signal Having the Constant Intensity Vector **n** With Parallel Planes $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane1}}$ and $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane2}}$ Having Normals **n**

Intersections of LT Signals In Planes Parallel To An Axis

We now want to derive a relationship between the intersections of $x(t_1, t_2, t_3)$ with parallel planes that are normal to one of the axes.

Consider two parallel planes, $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane1}}$ and $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane2}}$, having a normal $\mathbf{n_1}$ *that points along the* t_3 *axis* and therefore is given by $\mathbf{n_1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, as shown in Figure (3.2). Let $x(t_1, t_2, t_3)$ be a LT signal, where the constant signal vector $\mathbf{n}$ is generally different from $\mathbf{n_1}$, and denote the 3-tuple (t_1, t_2, t_3) by the point A in the plane $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane1}}$, as shown in Figure (3.2). Further, let the 3-tuple B lie in the plane $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane2}}$ such that *the* *line AB is in the direction of the constant signal vector* $\mathbf{n}$, as shown in Figure (3.2), implying that the signal at B has the same value as the signal at A.

FIGURE 3.2 A Rectangular Region Enclosing a 3D LT Signal

Consider the 3D rectangle [$A;B$] in Figure (3.2) and let the two planes $\mathbf{R}^3_{\mathbf{plane1}}$ and $\mathbf{R}^3_{\mathbf{plane2}}$ be perpendicular distance Δt_3 apart, as shown. Then it follows directly from the geometry of this rectangle in Figure (3.2) that the 3-tuple B is given by

$$B = \left(t_1 + \Delta t_3 \frac{n_1}{n_3}, t_2 + \Delta t_3 \frac{n_2}{n_3}, t_3 + \Delta t_3\right) \tag{3.2}$$

However, the signal has the same value at B as at A, so $x(B) = x(A)$ and therefore, from equation (3.2),

$$x\left(t_1 + \Delta t_3 \frac{n_1}{n_3}, t_2 + \Delta t_3 \frac{n_2}{n_3}, t_3 + \Delta t_3\right) = x(t_1, t_2, t_3) \tag{3.3}$$

which is the required result.

Equation (3.3) implies that, if we slice the LT signal in successive planes that are normal to the t_3 axis and Δt_3 apart, then the signals in each of these successive planes are identical except that *they undergo a shift from plane to plane that is given by* $\Delta t_3(n_1 / n_3)$ *in the direction of the* t_1 *axis and* $\Delta t_3(n_2 / n_3)$ *in the direction of the* t_2 *axis*. This corresponds to the 2D shift vector $\mathbf{s_3}$ where

$$\mathbf{s_3} \equiv [AC, CG]^T = \left[(\Delta t_3)\frac{n_1}{n_3}, (\Delta t_3)\frac{n_2}{n_3}\right]^{\mathbf{T}} \tag{3.4}$$

where *AC* and *CG* are shown.

The Spatio-Temporal Case: If t_3 is the temporal dimension and t_1 and t_2 are spatial dimensions, then $\mathbf{s_3}$ describes the relative spatial displacement of the LT image signal over any time interval Δt_3. It follows that the 2D spatial **velocity vector** is given by

$$\mathbf{v_3} = \left[\frac{n_1}{n_3} \frac{n_2}{n_3}\right]^{\mathbf{T}} \tag{3.5}$$

This expression for the 2D spatial velocity vector in the 2D dynamic image is a useful practical measurement when designing 3D image-enhancement algorithms for 3D filters.

Example 10 A Typical 3D LT Movie Image Signal

Consider part of a typical continuous-domain movie scene in which an object signal $x(\mathbf{t})$, $\mathbf{t} \in \mathbf{R}^3$ moves with uniform velocity across the movie screen at an angle of 30 degrees to the horizontal in the NE direction, as shown in Figure (3.3) , and suppose that the object covers a distance on the screen equal to *20 cm* in *5 seconds*.

Find the 2D spatial shift $\mathbf{s_3}$ and the constant signal direction $\mathbf{n}$ of the 3D LT signal.

We have, $AC = 2\cos(30°)cm$. and $CG = 2\sin(30°)cm$. So, in equation ( ),

$$\mathbf{s_3} \equiv [2\cos(30°)\,\mathrm{cm}, 2\sin(30°)\,\mathrm{cm}]^{\mathbf{T}} = \left[5\frac{\mathrm{n}_1}{\mathrm{n}_3}\mathrm{cm}, 5\frac{\mathrm{n}_2}{\mathrm{n}_3}\mathrm{cm}\right]^{\mathbf{T}} \qquad (3.6)$$

and

$$\mathbf{n} = [(2\cos(30°)/5)n_3 cm \quad (2\sin(30°)/5)n_3 cm \quad n_3 sec]^T \qquad (3.7)$$

We note that the first two components of the vector $\mathbf{n}$ tells us the horizontal distance and vertical distance moved by the signal object in time n_3. If we *normalize* n_3 *to one second,* then

$$\mathbf{n} = [(2\cos(30°)/5)cm \quad (2\sin(30°)/5)cm \quad 1\ sec]^T \qquad (3.8)$$

Continuing this example by assuming that the 3D movie signal is uniformly sampled at 30 *pixels/cm*. along the t_1 and t_2 axes and at 1 frame every 1/60 second on the time axis t_3 , we arrive at the uniformly sampled version $\mathbf{x}(\mathbf{n}), \mathbf{n} \in \mathbf{Z}^3$, of this image. The corresponding direction of a constant signal vector in this discrete integer 3-tuple domain is given by the direction of the vector

$$\mathbf{n} = [(2\cos(30°)/5) \times 30 \text{ pixels } (2\sin(30°)/5) \times 30 \text{ pixels } 1 \times 60 \text{ pixels}]^T \qquad (3.9)$$

or, dividing each element of $\mathbf{n}$ by *60,*

$$\mathbf{n} = [(\cos(30°)/5) \text{ pixels } (\sin(30°)/5) \text{ pixels } 1 \text{ pixel}]^T \qquad (3.10)$$

Of course, we cannot expect that this direction $\mathbf{n}$ in $\mathbf{R}^3$ will cause sample points in successive temporal frames to coincide with $\mathbf{s_3}$-shifted versions of the *sampled* image $x(n_1, n_2, n_3)$. These sample points do, however, correspond to rectangular samples of $\mathbf{s_3}$-shifted versions of the *continuous-domain LT* image $x(t_1, t_2, t_3)$.

FIGURE 3.3 An Example Of A LT Television Signal in $\mathbf{R}^3$

3.1.2 MD PLANE WAVE SIGNALS

Plane wave signals are very common. In the 2D case, plane waves $x(t_1, t_2)$ are often encountered in applications where a linear array of detectors measures the arrival of a signal from a distant source or target. Typical applications include seismic wave detection systems, linear array radio and sonar frequency directional detectors in radar and navigation systems, baseline array detector systems for astronomical imaging. The applications for 3D plane wave detection and enhancement are becoming more important as the processing speed and memory storage capability of modern signal processing systems is able to handle the large quantities of data. In the 3D case, seismic wavefront detection is now important, because it allows the direction of the seismic source to be more accurately determined than does 2D seismic processing. The potential for employing 3D plane waves in the compression, enhancement and transmission of television signals is of also of growing interest.

The MD Plane

We define a constant vector $\mathbf{d} \equiv [d_1, d_2, \ldots, d_d, \ldots, d_N]^{\mathbf{T}} \in \mathbf{R}^{\mathbf{N}}$ and consider the vectors $\mathbf{t} \equiv [t_1, t_2, \ldots, t_d, \ldots, t_N]^{\mathbf{T}} \in \mathbf{R}^{\mathbf{N}}$ that satisfy the equation

$$\mathbf{d}^{\mathbf{T}}\mathbf{t} \equiv d_1 t_1 + d_2 t_2 + \ldots d_d t_d + \ldots d_N t_N = l \tag{3.11}$$

where l is a scalar constant. The region in $\mathbf{R}^{\mathbf{N}}$ where the above equation is satisfied is defined as the MD plane and denoted $\mathbf{R}^{\mathbf{N}}_{\mathbf{plane}}$. The vector $\mathbf{d}$ is defined as the **normal to the plane** $\mathbf{R}^{\mathbf{N}}_{\mathbf{plane}}$. If the length is unity, then $\mathbf{d}$ is referred to as the **unit normal**. One may express the functional dependence of this planar region on $\mathbf{d}$ and l by writing the region as $\mathbf{R}^{\mathbf{N}}_{\mathbf{plane}}(\mathbf{d}, l)$ where

$$\mathbf{R}^{\mathbf{N}}_{\mathbf{plane}}(\mathbf{d}, l) \equiv \{t_1, t_2, \ldots, t_d, \ldots, t_N | d_1 t_1 + d_2 t_2 + \ldots d_d t_d + \ldots d_N t_N = l\} \tag{3.12}$$

or, equivalently,

$$\mathbf{R}^{\mathbf{N}}_{\mathbf{plane}}(\mathbf{d}, l) \equiv \{ \mathbf{t} | \mathbf{d}^{\mathbf{T}}\mathbf{t} = l \} \tag{3.13}$$

Let **d** be a unit normal vector. Then the length l is the **projection of t in the direction of d** . In the 3D case, l has the geometric interpretation that it is the shortest distance between the 3D plane and the origin, as shown in Figure ?.??.

Plane Wave Signals

By considering all possible values of l in the interval $[-\infty,\infty]$, one generates an infinite number of MD planes having the same normal d.

The MD Plane Wave Signal

A MD signal $x(t)$ is a MD plane wave if there exists a MD vector **d** such that, for every l, $x(\mathbf{t})$ is constant everywhere in each of the planes $\mathbf{R}^{\mathbf{N}}_{\mathbf{plane}}(\mathbf{d}, l)$.

For a particular **d**, the *constant* value of $x(\mathbf{t})$ in each plane is only a function of l and may be written as $x_{\boldsymbol{plane}}(l)$. The plane wave signal can therefore be described in the alternate form

$$x_{plane}(l) | \mathbf{d}^{\mathbf{T}}\mathbf{t} = l \tag{3.14}$$

The unit vector in parallel with **d** is defined as the **propagation vector** of the MD wave and the direction of **d** is defined as the **direction of propagation** of the MD plane wave.

3.1.3 3D Plane Wave Signals

Two of the infinite number of 3D planes having the normal **d** are shown in Figure 2.22 . The signal $x(t_1, t_2, t_3)$ is constant in each of these planes, having value $x_{plane}(l_1)$ in the plane $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane}}(\mathbf{d}, l_1)$ and value $x_{plane}(l_2)$ in the plane $\mathbf{R}^{\mathbf{3}}_{\mathbf{plane}}(\mathbf{d}, l_2)$, as shown in the diagram. These planes have shortest distances l_1 and l_2 from the origin, respectively, as shown in Fig.2.??.

In general, there is a continuum of such planes, one for each l in $[-\infty,\infty]$. Over this range of l , the function $x_{plane}(l)$ describes the variation in the value of the plane wave as a function of the shortest distance of the 3D plane from the origin. Clearly, a 3D plane wave is completely described by the function $x_{plane}(l)$ and the vector **d**. It may be written as

$$x_{plane}(l) | d_1 t_1 + d_2 t_2 + d_3 t_3 = l \tag{3.15}$$

Example 11 A 3D Plane Wave Gate Signal

Consider the 3D plane wave signal $x_{plane}(l)\big|d_1t_1 + d_2t_2 + d_3t_3 = l$ defined by

$$\mathbf{d} \equiv [1, 1, 2]^T \tag{3.16}$$

$$x_{plane}(l) \equiv u^1(l - l_1) - u^1(l - l_2),\ l_1 < l_2 \tag{3.17}$$

where $u^1(l)$ is the previously defined 1D unit step function given by $u^1(l) \equiv 1,\ l \geq 0,\ \ u^1(l) \equiv 0,\ l < 0$. The corresponding 3D signal $x(t_1, t_2, t_3)$ is shown in Figure 2.23 from which it is observed that the signal is essentially a 3D unit amplitude pulse with a planar wavefront that is perpendicular distance l_2 from the origin. The pulse has thickness $(l_2 - l_1)$ and direction of propagation $[1, 1, 2]^T$. This example is typical of the envelope of a reflected radar or sonar pulse that is received on a 2D spatial array of detectors in t_1 and t_2 as a function of time t_3. Later, we shall be interested in signal processing algorithms that can selectively enhance such pulses on the basis of their direction **d**.

Example 12 3D Plane Wave Signals from 1D Signals by the Rotation of Coordinates

Consider the 1D signal $x_{plane}(t_1),\ t_1 \in \mathbf{R^1}$, as shown in Figure 2.??. This 1D signal may be used to describe the 3D plane wave signal

$$x_{plane}(t_1, t_2, t_3) \equiv x_{plane}(l)\big|1t_1 + 0t_2 + 0t_3 = l \tag{3.18}$$

This 3D plane wave $x(t_1, t_2, t_3)$ has the direction of propagation given by the unit vector $\mathbf{d} \equiv \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ which points along the t_1 axis. The constant signal planes are therefore perpendicular to the t_1 axis. Using the previous notation for a plane in $\mathbf{R^3}$, such planes may be written as $\mathbf{R^3}(\mathbf{d}, t_1)\big|\mathbf{d} \equiv \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T, -\infty \leq l \leq \infty$. The signal has value $x_{plane}(t_1)$ everywhere in the planes $\mathbf{R^3}(\mathbf{d}, t_1)$, as shown in Figure 2.??. Consider now the rotation of the coordinate system t by means of the rotations $\mathbf{R_1}$ and then $\mathbf{R_2}$, as shown in Figure 2.??, resulting in the new coordinate system u where

$$u = \mathbf{R_2R_1}t \tag{3.19}$$

Let us find the direction of propagation $\mathbf{d_u}$ of the plane wave in the new rotated coordinate system u. In the coordinate system t, the unit propagation vector $\mathbf{d} \equiv \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ and

$$\mathbf{d_u} = \mathbf{R_2R_1d} \tag{3.20}$$

Substituting for **d** and using the expression in Example 17? in the above equation gives

$$\mathbf{d}_u = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \cos\theta_2\sin\theta_1 & \cos\theta_2\cos\theta_1 & -\sin\theta_2 \\ \sin\theta_2\sin\theta_1 & \sin\theta_2\cos\theta_1 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 \\ \cos\theta_2\sin\theta_1 \\ \sin\theta_2\sin\theta_1 \end{bmatrix} \tag{3.21}$$

We have determined the propagation vector $\mathbf{d_u}$ of the plane wave in the new coordinate system $\mathbf{u}$. The variation of the signal in the direction of $\mathbf{d_u}$ is unaltered by the rotation and remains as $x_{plane}(l)$, where l is the distance from the signal plane to the origin in the new coordinate system. The equation of the plane wave in the new coordinate system is $x_{plane}(l)|\mathbf{d_u}^T\mathbf{u} = 0$.

Example 13 A 3D Impulse Plane Wave

As a special case of the 3D Plane Wave signal, consider the signal

$$x_{plane}(l) \equiv \delta^1(l - l_1)|\mathbf{d}^T\mathbf{t} = l \tag{3.22}$$

where $\delta^1(l)$ is the 1D unit impulse function. The signal $x(t_1, t_2, t_3)$ is zero everywhere outside of the plane $\mathbf{R}^3_{\mathbf{plane}}(\mathbf{d}, l_1)$. Everywhere inside the plane $\mathbf{R}^3_{\mathbf{plane}}(\mathbf{d}, l_1)$, the signal has infinite magnitude, as shown in Figure 2.24. Find the 3D integral of this signal in a 3D volume that completely encloses the plane $\mathbf{R}^3_{\mathbf{plane}}(\mathbf{d}, l_1)$. (Hint: refer back to the definition of the 1D impulse function and consider first the 3D integral in a finite size box that encloses some finite area A of the plane.)

3.1.4 Plane Waves as a Subset of LT Signals

Consider the general MD plane waves $\{x_{plane}(l)| = l, -\infty \le l \le \infty\}$ and, in particular, consider the MD plane $\mathbf{d}^T\mathbf{t} = 0$ that passes through the origin $\mathbf{0}$. Further, let $\mathbf{n}$ be a MD vector that is directed out from the origin $\mathbf{0}$ such that it lies in this plane; that is, such that $\mathbf{d} \cdot \mathbf{n} = 0$. Any such vector $\mathbf{n}$ is a constant signal intensity vector and therefore **the general MD plane wave signal is a special case of a LT MD signal**. For N>1 it is possible to find an infinite number of constant signal unit vectors $\mathbf{n}$ that satisfy $\mathbf{d} \cdot \mathbf{n} = 0$. For example, in the case of 3D Plane Waves, there is clearly an infinite number of constant signal unit vectors $\mathbf{n}$ that lie in the plane, each pointing in a different direction, as shown in Figure 2.22.

3.1.5 A 3D Sinusoidal Plane Wave

Consider the 3D signal

$$x(\mathbf{t}) \equiv \sin(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3), \mathbf{t} \in \mathbf{R}^3 \tag{3.23}$$

This signal is constant and equal to $\sin(l)$ in the infinite number of parallel 3D planes given by

$$\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3 = \omega^T \mathbf{t} = l \qquad -\infty \le l \le \infty \tag{3.24}$$

Therefore, the signal $\sin(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3)$ *is a 3D plane wave* for which

$$x_{plane}(l) = \sin(l), \qquad -\infty \le l \le \infty \tag{3.25}$$

This plane wave may therefore be written as $\sin(l)|_{\omega^T \mathbf{t}}$ where the 3D direction of the propagation unit vector $\mathbf{d}$, which is normal to the 3D planes, is given by

$$\mathbf{d} = \pm\left[\frac{\omega_1}{\|\omega\|_2}\ \frac{\omega_2}{\|\omega\|_2}\ \frac{\omega_3}{\|\omega\|_2}\right]^T \tag{3.26}$$

Consider a vector $\mathbf{n} = [n_1\ n_2\ n_3]^T$, directed from the origin, and consider also the particular 3D plane $\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3 = 0$ that passes through the origin $\mathbf{0}$. The vector $\mathbf{n}$ is a constant signal vector if it lies in this 3D plane; that is, if it satisfies the condition $\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 = 0$ or, eqivalently, the condition $\omega \cdot \mathbf{n} = 0$. For any given values of ω_1, ω_2 and ω_3, there is an infinite number of different 3-tuples that satisfy $\omega \cdot \mathbf{n} = 0$. Therefore, it is always possible to find an infinite number of constant signal vectors $\mathbf{n}$ and consequently the signal $\sin(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3)$ *is an LT signal*.

3.1.6 MD Uniplanar Signals

A signal $x(\mathbf{t})$ is a MD **uniplanar signal** if it is zero everywhere outside of a hyperplane region $\mathbf{R}^{\mathbf{M}}_{\mathbf{plane}} \subset \mathbf{R}^{\mathbf{M}}$ that was defined in equation ( ). That is, $x(\mathbf{t})$ is a MD uniplanar signal if

$$x(\mathbf{t}) \equiv 0, \qquad \mathbf{t} \notin \mathbf{R}^{\mathbf{N}}_{\mathbf{plane}} \tag{3.27}$$

We will find that uniplanar signals are encountered in the frequency domain representation of LT signals.

Example 14 3D Uniplanar Signals

A 3D *uniplanar* signal is shown in Figure 2.24. It satisfies

$$x(t_1, t_2, t_3) \equiv 0, \qquad \mathbf{t} \notin \mathbf{R}^{\mathbf{M}}_{\mathbf{plane}} \tag{3.28}$$

where

$$\mathbf{R}^{\mathbf{M}}_{\mathbf{plane}} \equiv \{t_1, t_2, t_3 | a_1 t_1 + a_2 t_2 + a_3 t_3 = l\}$$

Equivalently, the signal has the region of support $\mathbf{R}^{\mathbf{M}}_{\mathbf{plane}}$. Its value in the plane is not generally constrained.

A MD uniplanar signal is a MD plane wave if it is constant in the region $\mathbf{R}^{\mathbf{M}}_{\mathbf{plane}}$, as shown in the following example.

Example 15 MD Uniplanar Plane Waves

Consider a MD uniplanar plane wave signal that is defined as zero except in a particular MD plane where it is equal to the (possibly complex) constant z_o . This signal may therefore be written in terms of the unit impulse operator δ_1 as

$$z_o \delta_1(l) | \omega^{T} t = l \tag{3.29}$$

This is a signal that lies in the plane, as shown in Figure 2.24, and which generally may equal *any complex constant value* z_o in the plane.