CHAPTER 4 Energy and Power

4.1 THE POWER OF SIGNALS

The concepts of signal power and signal energy are of increasing importance in the design of both continuous and discrete domain systems.

4.1.1 Complex Power

Continuous-Domain Case:

The **complex power** $p(\mathbf{t})$ **of a pair of complex signals** $x(\mathbf{t})$ and $y(\mathbf{t})$, having the same domains, is defined as

$$p(\mathbf{t}) \equiv y^*(\mathbf{t})x(\mathbf{t}) \tag{4.1}$$

where the superscript * implies complex conjugation. The domain of $p(\mathbf{t})$ is therefore the same as the domain of $x(\mathbf{t})$ and $y(\mathbf{t})$. The MD complex power is itself a MD complex signal.

It is often the case that the signals $x(\mathbf{t})$ and $y(\mathbf{t})$ are the same signal, in which case we define **the power of the complex signa**l $x(\mathbf{t})$ as

$$p(\mathbf{t}) \equiv x^*(\mathbf{t})x(\mathbf{t}) \; = \; |x(\mathbf{t})|^2 \tag{4.2}$$

which is clearly non-negative and real. The above definitions of power apply to continuous-domain, discrete-domain and mixed-domain MD signals.

Discrete-Domain Case:

Often, the discrete numbers t_k, for each dimension k, are the set of integers $\mathbf{Z^N}$ which we write as n_k. Then the corresponding expressions for the complex power of pairs of signal sequences, defined on integer N-tuples, is

$$p(n_1\ n_2\ n_3\ \dots\ n_N) \equiv y^*(n_1\ n_2\ n_3\ \dots\ n_N)x(n_1\ n_2\ n_3\ \dots\ n_N) \qquad (4.3)$$

which is written

$$p(\mathbf{n}) \equiv y(\mathbf{n})^* x(\mathbf{n}), \quad \mathbf{n} \in \mathbf{Z^N} \qquad (4.4)$$

and therefore, for a single signal,

$$p(\mathbf{n}) \equiv x(\mathbf{n})^* x(\mathbf{n}) = |x(\mathbf{n})|^2 \qquad (4.5)$$

4.1.2 Real Power

The **real power** of a pair of real signals $x(\mathbf{t})$ and $y(\mathbf{t})$ is a special case of equations (4.4) and (4.5) and given by

$$p(\mathbf{t}) \equiv y(\mathbf{t})x(\mathbf{t}), \quad \mathbf{t} \in \mathbf{R^N} \qquad (4.6)$$

and the real power of a single real signal is given by

$$p(\mathbf{t}) \equiv x(\mathbf{t})^2, \quad \mathbf{t} \in \mathbf{R^N} \qquad (4.7)$$

For the discrete-domain case, where the signal $x(\mathbf{n})$ is defined on the integer N-tuples, the real power of a pair of real signals is given by

$$p(\mathbf{n}) \equiv y(\mathbf{n})x(\mathbf{n}), \quad \mathbf{n} \in \mathbf{Z^N} \qquad (4.8)$$

and the real power of a single signal is given by

$$p(\mathbf{n}) \equiv x(\mathbf{n})^2, \quad \mathbf{n} \in \mathbf{Z^N} \qquad (4.9)$$

4.2 THE ENERGY OF SIGNALS

Having defined the concept of power at location **t** or **n** for a MD signal (or pair of signals), we now consider the related concept of the MD energy of a signal (or pair of signals).

4.2.1 Energy of Continuous-Domain Signals

The **energy of a pair of continuous-domain MD signals or of a single MD continuous-domain signal** at the MD N-tuple $\mathbf{t_o} = \{t_{1o}, t_{2o}, t_{3o}, \ldots, t_{ko}, \ldots, t_{No}\}$ is defined in terms of the signal power by

$$E(\mathbf{t_o}) = \int_{t_1 = -\infty}^{t_{1o}} \int_{t_2 = -\infty}^{t_{2o}} \cdots \int_{t_k = -\infty}^{t_{ko}} \cdots \int_{t_m = -\infty}^{t_{mo}} p(t) dt_m \ldots dt_k \ldots dt_2 dt_1 \qquad (4.10)$$

which is written for brevity as

$$E(\mathbf{t_o}) = \int_{-\infty}^{t_o} p(\mathbf{t}) d\mathbf{t} \qquad (4.11)$$

Signal energy is therefore the integration of power in the MD space over $\mathbf{R^N_{causal}}$ where

$$\mathbf{R^N_{causal}} \equiv \bigcap_{k=1}^{\mathbf{N}} \mathbf{R_k} \qquad (4.12)$$

and where $\mathbf{R_k}$ is the one dimensional interval

$$\mathbf{R_k} \equiv \{t_k \in R^1 | t_k \le t_{ko}\} \qquad (4.13)$$

It is possible to select other regions in MD space over which power is integrated in order to arrive at alternate definitions of MD power. The region $\mathbf{R^N_{causal}}$ is chosen because it corresponds to many MD physical systems in which previous processing of the signal $x(\mathbf{t_o})$ has only occurred in the region $\mathbf{R^N_{causal}}$. In this case, the signals at $\mathbf{t_o}$ are *caused* only by events that have occurred in $\mathbf{R^N_{causal}}$. The regions $\mathbf{R^1_{causal}}$ and $\mathbf{R^2_{causal}}$ are shown in Figure (4.1)

FIGURE 4.1 The Regions $\mathbf{R^1_{causal}}$ and $\mathbf{R^2_{causal}}$.

4.2.2 Energy of Discrete-Domain Signals

The **energy of a pair of discrete-domain MD signals or of a single MD discrete signal** at the mD N-tuple $\mathbf{n_o} = \{n_{1o}, n_{2o}, n_{3o}, \ldots, n_{ko}, \ldots, n_{No}\}$ is defined as

$$E(\mathbf{n_o}) = \sum_{\mathbf{n} = -¥}^{\mathbf{n_o}} p(\mathbf{n}) \tag{4.14}$$

where

$$\sum_{\mathbf{n} = -\infty}^{\mathbf{n_o}} \equiv \sum_{n_1 = -¥}^{n_{1o}} \sum_{n_2 = -¥}^{n_{2o}} ¼ \sum_{n_k = -¥}^{n_{ko}} ¼ \sum_{n_N = -¥}^{n_{No}} \tag{4.15}$$

This definition for the energy of discrete-domain signals is similar to that for continuous-domain signals except that integration is replaced by summation. The region $\mathbf{R^N_{causal}}$ over which the summation is performed is

$$\mathbf{R^N_{causal}} \equiv n_k \big| n_k \le n_{ko} \tag{4.16}$$

It follows from equations (4.4) and (4.14) that a pair of *complex* discrete-domain MD signals has energy at $\mathbf{n_o}$ given by

$$E(\mathbf{n_o}) = \sum_{\mathbf{n} = -\infty}^{\mathbf{n_o}} y(\mathbf{n})^* x(\mathbf{n}) \tag{4.17}$$

and a single *complex* discrete domain MD signal $x(\mathbf{n})$ has energy at $\mathbf{n_o}$ given by

$$E(\mathbf{n_o}) = \sum_{\mathbf{n} = -\infty}^{\mathbf{n_o}} |x(\mathbf{n})|^2 \tag{4.18}$$

For *real* signals, the conjugation operator and the magnitude operator are not required in equations (4.17) and (4.18), respectively.

4.2.3 Finite Energy Signals

A continuous-domain signal, or pair of signals, is **finite energy** iff

$$E(\mathbf{t}) < \infty, \ \forall \mathbf{t} \in \mathbf{R}^{\mathbf{N}} \tag{4.19}$$

That is, the energy must be finite everywhere in the entire MD space. Similarly, a discrete-domain signal, or signal pair, is finite energy iff

$$E(\mathbf{n}) < \infty, \ \forall \mathbf{n} \in \mathbf{Z}^{\mathbf{N}} \tag{4.20}$$

Finite energy signals play an important role in the analysis of signal processing systems, especially when considering the stability of signal processing circuits and algorithms or the frequency domain performance of systems. Finite energy signals are sometimes referred to as **bounded energy** signals.

The Continuous-Domain Case

According to equations (4.2) and (4.11), an infinite duration signal $x(\mathbf{t})$ is a finite energy signal iff

$$E(\mathbf{t_0}) \equiv \int_{-\infty}^{\mathbf{t_0}} |x(\mathbf{t})|^2 d\mathbf{t} < \infty, \ \forall \mathbf{t_0} \in \mathbf{R}^{\mathbf{N}} \tag{4.21}$$

The non-negative integrand $|x(\mathbf{t})|^{\mathbf{2}}$ in equation (4.21) ensures that

$$E(\mathbf{t_0}) \le E(\infty), \ \forall \mathbf{t_0} \in \mathbf{R}^{\mathbf{N}} \tag{4.22}$$

and that the *upper bound* of the left side of equation (4.22) *is equal to* $E(\infty)$. Therefore, the left side of equation *may be replaced by its upper bound* and we have the following necessary and sufficient condition (equivalent to equation (4.21)) for a continuous-domain MD signal to be finite energy

$$E(\infty) \equiv \int_{-\infty}^{\infty} |x(\mathbf{t})|^2 d\mathbf{t} < ¥ \tag{4.23}$$

A continuous-domain *infinite duration* MD signal that satisfies equation (4.23) is said to be *square integrable* (SI). Clearly, continuous-domain MD signals, or signal pairs, are finite energy only if they are SI. Finite energy implies the SI property which in turn implies that continuous-domain signals, or signal pairs, are amplitude-bounded.

MD Signals That Approach Zero Asymptotically At All Infinite Distances in $\mathbf{R}^N$

Suppose that the signal $x(\mathbf{t})$ is of infinite duration and its value approaches zero as **t** approaches *all* points that are infinitely distant from the origin **t=0** . An example is shown in Figure (4.2) for the 3D case. Defining the MD Euclidean distance of **t** from the origin **0** as

$$R \equiv \|\boldsymbol{t}\|_2 \tag{4.24}$$

then an MD signal is asymptotic to zero at all infinitely distant points in its domain if

$$\lim_{R \to \infty} x(\mathbf{t}) = 0 \tag{4.25}$$

for all directions in which **t** *is taken to an infinite distance from the origin.* It may be shown that the limit in equation (4.25) is a necessary *but not sufficient* condition to ensure the SI property (and therefore the finite energy property).

FIGURE 4.2 An 3D Signal That Is Asymptotic To Zero

The Discrete Domain Case

For discrete-domain MD signals, a similar situation applies. The condition for finite energy of a discrete domain MD signal is

$$E(\mathbf{n_0}) \equiv \sum_{\mathbf{n}=-\infty}^{\mathbf{n_0}} |x(\mathbf{n})|^2 \tag{4.26}$$

which is also equivalent to the condition

$$E(\infty) < \infty \tag{4.27}$$

A signal that satisfies the above constraint is said to be square summable (SS). All discrete-domain finite energy signals, or signal pairs, are amplitude bounded and all finite duration amplitude bounded discrete domain signals, or signal pairs, have finite energy. However, infinite duration amplitude bounded discrete domain signals are finite energy only if they are SS.

In summary, and for the contiuous-domain case case,

$$SquareIntegrable \Leftrightarrow FiniteEnergy$$

$$FiniteEnergy \Rightarrow AmplitudeBounded$$

$$FiniteEnergy \Rightarrow \lim_{R \to \infty} x(\mathbf{t}) = 0$$

$$FiniteDurationAmplitudeBounded \Rightarrow FiniteEnergy$$

