CHAPTER 5 The Fourier Series

So far, MD signals $x(\mathbf{t})$ have only been considered over their domain *dmn* **t**. In this chapter, it is shown that signals may usefully be described in another domain; the frequency domain. The methods of Fourier analysis are used to transform a signal from *dmn* **t** to the frequency domain. It will be shown that signals and signal processing systems may be designed in terms of frequency domain parameters and frequency domain criteria.

In Section 3.1, the Fourier Series of 1D periodic signals is introduced, beginning with the continuous-domain case for which it is shown that such signals may be completely represented as the sum of 1D sinusoids. The Exponential Form of the Fourier Series is defined and employed to show that this class of signals may be represented as a sum of periodic exponential signals, known as **phasors**. By means of the phasor representation, the frequency domain is introduced, including the concept of the spectrum of a signal. The relationship between the power spectrum and the average power of 1D periodic continuous domain signal is established. All of these results are extended, in a natural way, to the discrete domain.

In Section 3.2, the Fourier Series is extended to include MD periodic signals, with emphasis on discrete-domain signals. The concepts of harmonics, phasors and the frequency-domain are introduced for the case of MD signals. It is shown that MD periodic signals may be represented as the sum of MD phasors.

In Section 3.3, the MD Fourier Transform is introduced for the purpose of describing non-periodic continuous-domain MD signals in the frequency domain. The MD version of Parseval's Theorem is introduced and used to define the MD Energy Density Spectrum. The connection between the energy representations of a MD signal in the two domains, *dmn* **t** and *dmn* ω , is established.

The Linear Transformation Property of the MD Fourier Transform is developed and used to establish relationships between operations on the signal in *dmn* **t** and corresponding operations in the frequency domain *dmn* ω . For example, such operations as MD scalings, rotations and shifts are considered in both domains. Some of the symmetry properties of MD signals are also considered in both domains

5.1 THE 1D FOURIER SERIES OF CONTINUOUS-DOMAIN SIGNALS

The Fourier Series representation of a signal is defined differently for the continuous-domain and the discrete-domain cases. It will be shown that the Fourier Series allows periodic signals to be exactly represented everywhere on the real line, whereas non-periodic signals may only be represented on some finite interval on the real line. In spite of this restriction, the Fourier Series forms the basis for development of the Discrete Fourier Transform and other important discrete transform methods.

Consider a real amplitude-bounded continuous domain 1D periodic signal $x^\circ(t)$ having periodicity T_p. (The superscript $\circ$ is used throughout to indicate that a signal or function is defined to be periodic over its domain). Thus, $x^\circ(t) = x^\circ(t - rT_p)$, $\forall r \in \mathbf{Z}^1$. The trigonometric Fourier series representation of such a signal is given by

$$x^\circ(t) = a_0 + \sum_{k=1}^{k=\infty} [a_k \cos(2\pi k f_p t) + b_k \sin(2\pi k f_p t)],\ k \in \mathbf{N}^1 \tag{5.1}$$

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where a_0, a_k and b_k are real constants and

$$f_p \equiv \frac{1}{T_p} \tag{5.2}$$

We refer to f_p as the fundamental frequency of the signal $x^\circ(t)$. The Fourier Series implies that a 1D continuous-time periodic signal may be exactly represented by the (possibly infinite) sum of cosinusoidal signals $a_k \cos(2\pi k f_p t)$ and sinusoidal signals $b_k \sin(2\pi k f_p t)$. This equivalence is illustrated in Figure (5.1) where the cosinusoidal terms are shown for $k = 0, 1, 2$ and the sinusoidal terms for $k = 1, 2$.

FIGURE 5.1 Representation of a 1D Signal as the Sum of its Fourier Components

(insert diagran here)

For any particular $x^{\circ}(t)$, there exists a unique set of real constants a_0, a_k and b_k , known as the **trigonometric Fourier coefficients** of $x^{\circ}(t)$, that determines the amplitudes of the sinusoidal and cosinusoidal terms in the Fourier Series representation. The terms $a_k \cos(2\pi k f_p t)$ and $b_k \sin(2\pi k f_p t)$ are referred to as the **harmonics** of the periodic signal $x^{\circ}(t)$. The harmonics have frequencies kf_p that are uniformly spaced at integer multiples k of the fundamental frequency f_p . Taken as a set, we refer to the harmonics as the **frequency spectrum** of $x^{\circ}(t)$.

It is possible (see Problem 3.??) to find the trigonometric Fourier coefficients a_0, a_k and b_k as functions of $x^{\circ}(t)$ by integrating over one period of t, the three terms, $x^{\circ}(t)$, $\tilde{x}(t)\cos(2\pi k f_o t)$ and $\tilde{x}(t)\sin(2\pi k f_o t)$. This leads directly to the following expressions for the Fourier Series coefficients

$$a_o = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x^{\circ}(t) dt \tag{5.3}$$

$$a_k = \frac{1}{2T_p} \int_{-T_p/2}^{T_p/2} x^{\circ}(t) \cos(2\pi k f_p t) dt \tag{5.4}$$

and

$$b_k = \frac{1}{2T_p}\int_{-T_p/2}^{T_p/2} x°(t)\sin(2\pi k f_p t)dt,\ (k \in N^1) \tag{5.5}$$

Convergence of the Fourier Series

It turns out that there are certain mathematical constraints on $x°(t)$ that must be met for the Fourier Series in (5.1) to be valid everywhere in t. The right side of equation (5.1) converges to $x°(t)$, the left side, for all amplitude-bounded $x°(t)$ at all points where $x°(t)$ is continuous(-valued) *provided that the number of amplitude discontinuities is finite*. At points t_o on t where $x°(t)$ is not continuous(-valued), the Fourier Series *converges to the average of the right and left hand limits* of $x°(t)$; that is, *to the average* of $x°(t_o^-)$ and $x°(t_o^+)$.

From a practical point of view, all *physically observable periodic signals* possess a valid Fourier series; that is, one that converges everywhere in *t*. This is because all physical signals must have a finite number of amplitude discontinuities. However, signals that have an infinite number of such discontinuities can be constructed mathematically (see Problem 3.??).

5.1.1 The Exponential Form of the Fourier Series

Writing the *k th* sinusoidal and cosinusoidal harmonics in equation ( ) in terms of exponentials, according to equations ( ), gives

$$x°(t) = a_0 + \sum_{k=1}^{k=\infty}\left[\frac{a_k}{2}[e^{j2\pi k f_p t} + e^{-j2\pi k f_p t}] + \frac{b_k}{2j}[e^{j2\pi k f_p t} - e^{-j2\pi k f_p t}]\right] \tag{5.6}$$

Note that we have chosen to express real sinusoidal and cosinusoidal signals as complex exponential signals. This step is very important and is widely used. It leads to a more straightforward approach to the analysis of signal processing systems than if cosine and sine terms are retained. (The primary reason for the simplification is that *exponential functions remain as exponential functions when multiplied together*, allowing more simple expressions than the complicated trigonometric identities that are required to simplify expressions containing products of sine and cosine terms).

There are two types of exponential terms in the above expression. First, the term $exp(j2\pi k f_p t)$ is a periodic continuous time *complex* 1D signal of the type considered in section 2.?? . This term is shown in Figure 3.2 as **a complex number that rotates, as a function of t, on a circle of unit radius in the anti-clockwise direction at a uniform rate of k revolutions per period $T = 1/f_p$.**

FIGURE 5.2 The Clockwise-Rotating and Anti-clockwise Roating 1D Phasors

(insert here)

The second type of exponential term is $exp(-j2\pi kf_p t)$, which is a similar complex number except that it rotates at the same rate but in **the clockwise direction**, also shown in Figure (5.2). These rotating exponentials are referred to as **phasors**.

The phasor $exp(j2\pi kf_p t)$, having anticlockwise rotation, is said to have **positive frequency** kf_p and the phasor $exp((-j)2\pi kf_p t$, having clockwise rotation, is said to have **negative frequency** $-kf_p$. Grouping the positive and negative frequency phasor terms in equation (5.6) gives

$$x^{\circ}(t) = a_0 + \sum_{k=1}^{k=\infty}\left[\frac{1}{2}(a_k - jb_k)(e^{j2\pi kf_p t}) + \frac{1}{2}(a_k + jb_k)(e^{-j2\pi kf_p t})\right], k \in \mathbf{N^1} \qquad (5.7)$$

The two terms inside the summation have imaginary components of opposite signs that cancel in the sum of these two terms; that is, the components in the direction of the imaginary axis of the phasors of positive and negative frequency are, for all t and each k, in equal and opposite directions, as illustrated in Figure (5.3).

FIGURE 5.3 Cancellation of the Imaginary Parts of the Complex Phasors

(insert here)

Defining

$$X_k \equiv \frac{1}{2}(a_k - jb_k),\ (k \in \mathbf{Z}^1) \quad \textbf{(5.8)}$$

equation (5.7) simplifies to the following alternate form of the Fourier Series

$$x^{\circ}(t) = \sum_{k=-\infty}^{k=\infty} X_k e^{j2\pi k f_p t},\ k \in \mathbf{Z}^1 \quad \textbf{(5.9)}$$

THE EXPONENTIAL FOURIER SERIES

where

$$X_0 \equiv a_0 \quad \textbf{(5.10)}$$

Note that the index of summation k is extended over all of the integers, including the negative integers. Equation (5.9) is the **Exponential Fourier series**. Note that the coefficients $X_k, k \neq 1$, are generally complex whereas the coefficients of the Trigonometric form of the Fourier Series are real.

It is easily shown from equations (5.8), (5.9) and (5.10) , (see Problem 3.??) , that the coefficients of the Exponential Fourier Series may be expressed in terms of $x^{\circ}(t)$ as

$$X_k \equiv \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x^{\circ}(t) e^{-j2\pi k f_p t} dt,\ (k \in \mathbf{Z}^1) \quad \textbf{(5.11)}$$

The set of coefficients X_k in equations (5.10) and (5.11), over all k , is referred to as the **spectrum** of $x^{\circ}(t)$. These coefficients X_k are often written in the polar form $M_k exp(j\theta_k)$

where

$$M_k = |X_k| \text{ and } \theta_k = arg(X_k) \quad \textbf{(5.12)}$$

In this case, the real discrete function M_k is referred to as the **magnitude line spectrum** of $x^{\circ}(t)$ and the real discrete function θ_k as the **phase spectrum** of $x^{\circ}(t)$. The relationships between signals and their corresponding spectra are of fundamental importance in signal processing.

So far, we have discussed the concept of a frequency spectrum of a narrow class of signals; that is, the continuous-domain 1D *periodic* signals. However, the concept extends to MD signals and also to discrete-domain and mixed-domain signals, as well as to non-periodic signals.

Many signal processing systems are designed on the basis of frequency domain criteria, including electronic filter circuits and filter algorithms, as well as image processing systems, control systems and adaptive systems. It is therefore worthwhile to explore further the concept of the frequency domain of a signal.

Example 16 The 1D Continuous-Time Periodic Gate Signal $g^{\circ}(t)$

Consider the continuous 1D function $\tilde{g}(t)$ in Figure 3.4 having periodicity T_p, pulse duration $2T_0$ and pulse amplitude A_0, which is often referred to as the gate function.

FIGURE 5.4 The 1D Continuous-Time Gate Signal $g^{\circ}(t)$

(insert here)

This type of waveform is very common in electronic engineering systems. It follows directly from equation (5.11) that the spectrum of this signal G_k is given by

$$G_k = \frac{1}{T_p}\int_{-T_0}^{T_0} A_0 e^{-j2\pi kt/T_p} dt$$

$$= \frac{A_0}{\pi k}[e^{-j2\pi kt/T_p}]_{-T_0}^{T_0}$$

$$= \frac{A_0}{\pi k}\left[\frac{e^{j2\pi kt/T_p} - e^{-j2\pi kt/T_p}}{2j}\right]$$

$$= \frac{A_0}{\pi k}[\sin(2\pi kT_0/T_p)]$$

$$= \frac{2A_0T_0/T_p}{2\pi kT_0/T_p}[\sin(2\pi kT_0/T_p)] \tag{5.13}$$

Substituting the fundamental radian frequency $\omega_p \equiv \frac{1}{T_p}$ into this equation gives the spectrum of $g^{\circ}(t)$ as

$$G_k = \frac{2A_0T_0}{T_p}\operatorname{sinc}(k\omega_pT_0), \qquad k \in \mathbf{Z}^1 \tag{5.14}$$

In this particular example, all of the G_k are real. This will not generally be the case but it does allow us to sketch the complete spectrum in a single diagram for this particular example, as shown in Figure (5.5). The positive values of G_k correspond to frequency regions where the phase spectrum θ_k is zero and the negative values of G_k correspond to frequency regions where θ_k is equal to π.

FIGURE 5.5 The Spectrum of the Gate Signal $g^{\circ}(t)$

(insert here)

Envelope of the Spectrum:

The spectrum G_k of $g^\circ(t)$ possesses a *sinc function envelope* having its first positive zero crossing where $k\omega_p T_0 = \pi$; that is, where

$$k = \frac{\pi}{\omega_p T_0} \tag{5.15}$$

which, in general, need *not* correspond to an *integer* value of k. Subsequent zero crossings of the *sinc* function envelope of G_k are at integer multiples of the frequency $k\omega_p$ *radians/unit* of t. Clearly, the zero crossings of the *sinc* envelope are not a function of the period T_p or of the pulse amplitude A_0, but only of the pulse duration . *The width of the envelope is in fact inversely proportional to the pulse duration* $2T_0$. The width of the envelope of the spectrum, often qualitatively referred to as the **bandwidth** of $g^\circ(t)$, increases as the pulse width becomes narrower.

Amplitude of the Spectrum:

Consider now the amplitude of the *sinc* envelope, which is proportional to

$$\frac{2A_0T_0}{T_p}$$

and therefore to pulse duration $2T_0$. Clearly, the amplitude of the *sinc* envelope decreases as the duration $2T_0$ is decreased.

Line Density of the Spectrum:

Consider the effect of increasing the period T_p with fixed pulse duration $2T_0$ and fixed pulse amplitude A_0. The amplitude of the envelope decreases in inverse proportion, without affecting its basic shape. The frequency distance between the lines (density of the harmonics) of the spectrum decreases and is equal to $\omega_p = 2\pi/T_p$ which is the fundamental periodic frequency. *The line spectrum becomes more dense as the period* T_p *increases.*

5.1.2 Some Properties of the 1D Fourier Series

It follows from equation (5.11) that, in general, the line spectrum $M_k \equiv |X_k|$ satisfies $M_k \equiv M_{-k}$. That is, **the magnitude spectrum is an even function of** k. It also follows from equation (5.11) that $\theta_k = \theta_{-k}$; that is, **the phase spectrum is an odd function of** k. Equivalently, X_k and X_{-k} are related via complex conjugation; that is,

$$X_{-k} = X_k^*, \forall k \in \mathbf{Z}^1 \tag{5.16}$$

According to equation (2.???) , the power of a periodic continuous domain *real 1D* signal is given by

$$p(t) = x^\circ(t)^2 \tag{5.17}$$

Clearly,the power $p(t)$ varies periodically with t. We now introduce the concept of the **average power** p_{av} of a periodic signal which is the average of $p(t)$ over any integer number of periods in t; clearly, p_{av} is constant. Thus

$$p_{av} \equiv \int_{-T_p/2}^{T_p/2} x^{\circ}(t)^2 dt = \int_{-T_p/2}^{T_p/2} \left[\sum_{k=-\infty}^{k=\infty} x_k e^{jk\omega_p t} \right]^2 dt \tag{5.18}$$

Consider now the evaluation of the above integrand;

$$\left[\sum_{k=-\infty}^{k=\infty} x_k e^{jk\omega_p t} \right]^2 = \sum_{k_1=-\infty}^{k_1=\infty} I_{k_1} \sum_{k_2=-\infty}^{k_2=\infty} I_{k_2} \tag{5.19}$$

where

$$I_k = e^{jk\omega_p t} \tag{5.20}$$

Then

$$I_{k_1} I_{k_2} = X_{k_1} e^{jk_1\omega_p t} X_{k_2} e^{jk_2\omega_p t} = X_{k_1} X_{k_2} e^{j(k_1+k_2)\omega_p t} \tag{5.21}$$

Multiplying out the individual terms in the double sum of equation (5.19) and substituting equation (5.21) gives

$$p_{av}(t) = \int_{-T/2}^{T/2} \sum_{k_1=-\infty}^{k_1=\infty} \sum_{k_2=-\infty}^{k_2=\infty} I_{k_1} I_{k_2} dt$$

$$= \sum_{k_1=-\infty}^{k_1=\infty} \sum_{k_2=-\infty}^{k_2=\infty} X_{k_1} X_{k_2} \int_{-T/2}^{T/2} e^{jk_1\omega_p t} e^{jk_2\omega_p t} dt \tag{5.22}$$

Consider the integral in the above equation. It is easily shown that

$$\int_{-T/2}^{T/2} e^{jk_1\omega_p t} e^{jk_2\omega_p t} dt = \begin{matrix} 0, \forall k_1, k_2 | k_1 \neq -k_2 \\ 1, \forall k_1, k_2 | k_1 = -k_2 \end{matrix} \tag{5.23}$$

Functions such as $exp(jk_1\omega_p t)$ and $exp(jk_2\omega_p t)$, having the above property under integration , are said to be **orthogonal** over the interval $[-T_p/2, T_p/2]$. Orthogonal functions play an important role in signal processing. The reason for their importance is illustrated here because the orthogonality of $exp(jk_1\omega_p t)$ and $exp(jk_2\omega_p t)$ in equation (5.23) allows equation (5.22) to be simplified to the following fundamentally important and simple expression for the power of a periodic signal in terms of its magnitude spectrum

$$p_{av}(t) = \sum_{k=-\infty}^{k=\infty} X_k X_{-k} = \sum_{k=-\infty}^{k=\infty} |X_k|^2 \quad \text{(5.24)}$$

That is, *the average power of a continuous time periodic signal $\tilde{x}^\circ(t)$ can be obtained by adding together the square of the magnitudes of the individual terms in the magnitude spectrum*. It is easily shown that the average power of each real k *th* harmonic

$$X_k exp(jk\omega_p t) + X_{-k} exp((-j)k\omega_p t) \quad \text{(5.25)}$$

is given by

$$p_{av_k} = |X_k|^2 + |X_{-k}|^2 = 2|X_k|^2 \quad \text{(5.26)}$$

This simple result follows from the orthogonality of the individual terms in the Fourier series. The line spectrum $|X_k|^2$ is referred to as the **power spectrum** of $x^\circ(t)$.

Other discrete transforms are used in signal processing, not all of which have orthogonal basis functions and, in such cases, it is generally not possible to infer an exact and simple relationship between the transform domain coeffcients and the power of the signal in the domain t. The reader will encounter such transforms in Chapter 4.

5.2 The 1D Fourier Series of Discrete Domain Periodic Signals

Consider a discrete domain real periodic 1D signal $x^\circ(nT)$, where $n = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, $T \in \mathbf{R^1}$, and where this signal is defined to have periodicity $N_p T, N_p \in \mathbf{N^1}$, as shown in Fig.(5.6).

FIGURE 5.6 A Real Discrete-Domain Periodic Signal $x^\circ(nT)$

(insert here)

This signal is therefore defined on uniform distances $n = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, $T \in \mathbf{R}^1$ and $x^{\circ}(nT) = x^{\circ}(nT - rN_pT)$, $\forall r \in \mathbf{Z}^1$. The periodic radian frequency is

$$\omega_p = 2\pi/(N_pT) \tag{5.27}$$

The Exponential Fourier series representation of $x^{\circ}(nT)$ is defeined as

$$x^{\circ}(nT) = \sum_{k=0}^{N_p-1} X(k)e^{jk\omega_p nT} \tag{5.28}$$

EXPONENTIAL FOURIER SERIES OF A DISCRETE DOMAIN SIGNAL

It can be shown, using the orthogonality of exponential signals under summation (see Problem 3.??), that the generally complex and periodic coefficients $X(k)$ are given by

$$X(k) = \frac{1}{N_p}\sum_{n=0}^{N_p-1} x^{\circ}(nT)e^{-jk\omega_p nT} \tag{5.29}$$

The coefficients $X(k)$ have periodicity N_p in k. Equation (5.29) implies that the signal $x^{\circ}(nT)$ may be expressed as a weighted sum of N_p discrete domain complex periodic exponential harmonic functions of the form $exp(jk\omega_p nT)$, which for each k, is a discrete phasor as shown in Fig.3.6. This representation of a discrete phasor is a set of points in the complex plane where the points are equidistant around the unit circle , rotating clockwise as a function of n. For each periodic increase N_pT in n, this phasor rotates by $-2\pi k$ radians (that is, by k revolutions of the origin).

FIGURE 5.7 The Discrete-Domain Phasor $exp(jk\omega_p nT)$

Complex Magnitude and Phase Spectra

The set of complex coefficients $X(k)$ is referred to as the **discrete complex frequency spectrum** of the discrete domain signal $\tilde{x}°(n)$. It follows that **a uniformly sampled periodic signal $\tilde{x}°(n)$ may be represented by the sum of N distinct harmonic complex phasors** $X(k)exp(-jk\omega_p n)$. The magnitude (or line) spectrum $M(k)$ and the phase spectrum $\theta(k)$ are given by

$$M(k) = |X(k)| \tag{5.30}$$

and

$$\theta(k) = arg|X(k)| \tag{5.31}$$

and the discrete power spectrum is defined as

$$|X(k)|^2 \tag{5.32}$$

DISCRETE POWER SPECTRUM

Average Power of Discrete Domain Periodic Signals:

The relationship between the average power in a *real* discrete domain 1D periodic sequence and the coefficients of its discrete Fourier series is similar to the continuous domain case. The average power p_{av} of a discrete domain periodic sequence is defined as

$$p_{av} \equiv \sum_{n=0}^{N-1} p(n) = \sum_{n=0}^{N-1} x^{\circ}(n)^2 \tag{5.33}$$

Substituting the discrete exponential Fourier series for $\tilde{x}(n)$ gives

$$p_{av} = \sum_{n=0}^{N-1} \left[\sum_{k=0}^{N-1} X(k) e^{-jk\omega_p nT} \right]^2 \tag{5.34}$$

Squaring the term in the brackets [ ] and using the following orthogonality condition for summed exponentials

$$\sum_{k=0}^{N-1} X(k_1) e^{-jk\omega_p n} X(k_2) e^{jk\omega_p n} = \begin{matrix} 1, k_1 = -k_2 \\ 0, k_1 \neq -k_2 \end{matrix} \tag{5.35}$$

leads directly to

$$p_{av} = \sum_{k=0}^{N-1} X(k) X(-k) = \sum_{k=0}^{N-1} |X(k)|^2 \tag{5.36}$$

Thus, it is also true in the discrete case that the total average power in a periodic signal is the sum of the squares of the magnitude spectrum. The term

$$|X(k)|^2 + |X(-k)|^2$$

is the average power of the individual *k th real* discrete harmonic; thus, the total average power is the sum of the average powers associated with each discrete periodic harmonic sequence.

Example 17 A Periodic Discrete Domain 1D Gate Signal

Consider a discrete domain periodic signal $\tilde{g}(nT)$ as shown in Figure (5.8) and consisting of samples of the 1D periodic gate signal that was considered in Example (16).

FIGURE 5.8 A Disrete-Domain Periodic 1D Gate Signal

(insert here)

The signal $g^{\circ}(nT)$ has periodicity N_pT and the sampled gate pulses have amplitude A_0 and duration $(2N_0+1)T$, as shown. The signal is illustrated in Figure (5.8) for the 'highly sampled' case where $N_p \equiv 100$, $N_0 \equiv 10$, $T \equiv 0.5$. (We make no attempt in this example to quantify the concept of a 'high' sample rate. The reader will have to wait until the Sampling Theorem is considered in Chapter 4.) In this case, $N_p \gg N_0 \gg 1$, implying that each gate pulse consists of many samples and that the period is much greater that the durations of the pulses. According to equation ( ), the exponential Fourier Series $G(k)$ is given by

$$G(k) = \sum_{n=0}^{N_p-1} g^{\circ}(nT)e^{-jk\omega_p nT} = \sum_{n=-N_0}^{n=N_0} A_0 e^{-jk\omega_p nT} \qquad (5.37)$$

Now, it is shown in Appendix A that

$$\sum_{n=-N_0}^{n=N_0} e^{-j\theta} = \frac{\sin\left[\frac{(2N_0+1)\theta}{2}\right]}{\sin\left[\frac{\theta}{2}\right]} \qquad (5.38)$$

Substituting equation (5.38) into equation (5.37), with $\theta = k\omega_p T$, gives

$$G(k) = \frac{A_0}{N_p}\frac{\sin\left[\frac{(2N_0+1)k\omega_p T}{2}\right]}{\sin\left[\frac{k\omega_p T}{2}\right]} = \frac{A_0}{N_p}\frac{\sin\left[\frac{(2N_0+1)\pi kT}{N_p}\right]}{\sin\left[\frac{\pi kT}{N_p}\right]}, (k \in \mathbf{Z}^1) \qquad (5.39)$$

This is the required result for the line spectrum $\tilde{G}(k)$ and is shown in Figure 3.9.

FIGURE 5.9 Spectrum $\tilde{G}(k)$ of the Discrete-Domain Periodci 1D Gate Signal $g(n)$

(insert here)

First, it is observed that $G(k)$ is defined on the entire set of integers in $\mathbf{Z}^1$. Observe that $G(k)$ is periodic in k with periodicity $1/N_p$. This makes good sense because the envelope of the line spectrum is expected to repeat at an interval that is determined by the repetition period N_p of the envelope of the signal $\tilde{g}(nT)$. The envelope of the main lobe of the line spectrum $G(k)$ crosses zero at $k = 1/N_0$ which again makes good sense because we expect the zero crossing to be determined by the 'duration' N_0 of the continuous domain version of the pulse. The distance between the lines of the line spectrum $G(k)$ is unity. The horizontal axis of this line spectrum can be cast in terms of radian frequency, rather than the index k of the harmonic, simply by scaling the axis by $2\pi/T$. This second scaled axis is also indicated in Figure (5.9) where it is observed that the envelope of $G(k)$ is periodic at the radian frequency $2\pi/N_pT$ and the lines are spaced at intervals of $2\pi/T$. The main lobe of the envelope then crosses zero at $2\pi/N_pT$ radians, which corresponds with the zero crossing of the main lobe of the spectrum of the continuous domain pulse considered in Example 3.1.

It may be shown that the line spectrum $G(k)$ of a discrete domain periodic signal $x^\circ(nT)$ is

- defined on lines at the radian frequency intervals $2\pi/T$
- has an envelope that has periodicity $2\pi/TN_p$
- the shape of the envelope, for the 'highly sampled' case, is similar to that of the spectrum of the corresponding continuous domain signal.

In general, the coefficients $G(k)$ of the discrete Fourier series exist on all $k \in \mathbf{Z}^1$ and are periodic with periodicity N_p, where N_p is the periodicity of $x^\circ(nT)$.

Repersentation of Nonperiodic Signal $x(t)$ on a Finite Interval by Periodic

Extension:

It is possible to employ the Fourier Series to represent a nonperiodic finite-duration signal $x(t)$ having a region of support on a finite interval $[0, L]$ in $dmn\, t$. This may be done by choosing a periodic signal $x^\circ(t)$ having a period equal to L and such that it is equal to the nonperiodic signal $x(t)$ over the period $[0, L]$. That is, by choosing $x^\circ(t)$ so that

$$x^\circ(t) = x(t), 0 \le t \le L \tag{5.40}$$

The function $x^\circ(nT)$ is then referred to as the **periodic extension** of the nonperiodic signal $x(t)$.

This technique for finding the frequency domain representation of a nonperiodic signal by periodic extension can and often does lead to significant practical problems. Unless the signal is of shorter duration than L, it is truncated to length L and there may then exist artificial edge discontinuities at the edges of the interval $[0, L]$, as shown in Figure(5.10).

FIGURE 5.10 Representation of a Signal on a Finite Interval

(insert here)

The assumption that $x(t)$ equals its periodic extension $x^\circ(t)$ is then no longer valid. We shall later find out that these edge effects are a most troublesome limitation of the Fourier Series approach that also apply to the Fourier Transform of discrete domain signals.

5.3 THE MULTIDIMENSIONAL FOURIER SERIES

The Fourier series representation of MD signals is basically a straightforward extension of the 1D case. We will leave the *continuous-domain* MD case to the exercises at the end of this chapter and consider here only the discrete domain MD case.

Consider the discrete-domain MD periodic signal that is defined on the domain of integer M-tuples as follows

$$x^{\circ}(n_1T_1, n_2T_2, \ldots, n_dT_d, \ldots, n_mT_m) =$$
$$x^{\circ}(n_1T_1 - rN_1T_1, n_2T_2 - rN_2T_2, \ldots, n_dT_d - rN_dT_d, \ldots, n_mT_m - rN_mT_m) \quad \textbf{(5.41)}$$

where , for each dimension d, the intersample distances are T_d and the period is N_dT_d with n_d and N_d positive integer constants. Then, writing

$$\mathbf{n} = \{n_1, n_2, n_3, \ldots, n_d, \ldots, n_m\} \quad \textbf{(5.42)}$$

and

$$\mathbf{k} = \{k_1, k_2, k_3, \ldots, k_d, \ldots, k_m\} \quad \textbf{(5.43)}$$

the discrete exponential Fourier series of $x^{\circ}(n)$ is given by

$$x^{\circ}(\mathbf{n}) = \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\ldots\sum_{k_m=0}^{N_m-1} X(\mathbf{k})exp\left[j\sum_{d=1}^{m} k_d\omega_d n_d T_d\right] \quad \textbf{(5.44)}$$

5.3.1 MD Complex Phasors and MD Harmonics

The MD complex exponential term

$$exp\left[j\sum_{d=1}^{m} k_d\omega_d T_d\right] \quad \textbf{(5.45)}$$

will be considered many times during the study of MD systems. It is a discrete domain periodic exponential **MD phasor** with MD periodicity **N** where

$$N \equiv \{N_1, N_2, \ldots, N_d, \ldots, N_m\} \quad \textbf{(5.46)}$$

(The MD phasor is the theoretical input signal that is applied to a MD system in order to calculate its steady state frequency response).

The set of complex MD discrete domain exponential Fourier series coefficients $X(k_1, k_2, \ldots, k_m)$, which we write as $X(\mathbf{k})$, is the frequency spectrum of the MD discrete domain signal $x^{\circ}(\mathbf{n})$ and can be obtained from $x^{\circ}(\mathbf{n})$ according to the expression

$$X(\mathbf{k}) = \frac{1}{N_1 N_2 \ldots N_m} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \ldots \sum_{k_m=0}^{N_m-1} x^\circ(\mathbf{n}) exp\left[-j \sum_{d=1}^{m} k_d \omega_d n_d T_d\right] \quad \textbf{(5.47)}$$

Inspection of the above equation reveals that *the spectrum* $X(\mathbf{k})$ *has the same MD periodicity as* $x^\circ(\mathbf{n})$; that is, it has period **N** and therefore has periodicity N_d in each of its dimensions $d = \{1, 2, 3, \ldots, m\}$.

It follows that a uniformly sampled periodic MD signal $x^\circ(\mathbf{n})$ may be represented by the sum , that is the superposition of, $(N_1 N_2 N_3 \ldots N_m)$ distinct harmonic complex phasors , each of which has the general form

$$X(k_1, k_2, k_3, \ldots, k_m) exp\left[j \sum_{d=1}^{m} \frac{2\pi}{N_d} n_d k_d\right] \quad \textbf{(5.48)}$$

This is a fundamemtally important result because it allows us to anticipate how we might try to design MD systems to spectrally filter (that is enhance) MD signals by attempting to remove some of these harmonics. So far, we have not considered the type of signal processing that would accomplish such frequency domain filtering. Some additional mathematical tools are required. For example, the MD signals that we wish to process are usually duration bounded and therefore non-periodic; we clearly need some additional techniques to describe the frequency domain content of such signals.

Example 18 The 2D Discrete-Domain Periodic Gate Function

We have previously considered the discrete-domain periodic 1D gate function. Consider now the 2D version $g^\circ(n_1, n_2)$ shown in Fig.3.??10??, which is defined to have integer duration N_{01} in the direction of n_1 and integer duration N_{02} in the direction of n_2 , periodicities N_{p1} and N_{p1} in the respective directions and amplitude A_0, as shown.

FIGURE 5.11 The 2D Discrete-Domain Periodic Gate Function $g^\circ(n_1, n_2)$

According to equation (5.44), the Fourier series of this discrete domain signal is given by

$$G(k_1, k_2) = \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} g^{\circ}(\mathbf{n}) exp\left(-j\frac{2\pi}{N_{01}}k_1 n_1 - j\frac{2\pi}{N_{02}}k_2 n_2\right)$$

$$= \sum_{k_1=-N_{01}/2}^{N_{01}/2} A_0 exp\left(-j\frac{2\pi}{N_{01}}k_1 n_1\right) \sum_{k_2=-N_{02}/2}^{N_{02}/2} A_0 exp\left(-j\frac{2\pi}{N_{02}}k_2 n_2\right) \quad (5.49)$$

Using the result in Appendix A to evaluate the two summations in the above equation gives

$$G(k_1, k_2) = \frac{A_0}{N_{01}N_{02}}\left(\frac{\sin\left(\frac{(2N_{01}+1)\pi k_1 T}{N_{p1}}\right)}{\sin\left(\frac{\pi k_1 T}{N_{p1}}\right)}\right)\left(\frac{\sin\left(\frac{(2N_{02}+1)\pi k_2 T}{N_{p1}}\right)}{\sin\left(\frac{\pi k_2 T}{N_{p1}}\right)}\right) \quad (5.50)$$

which is the required result and is shown in Figure (5.12) for the particular numerical values

$$A_0 \equiv 50,\ T \equiv 1,\ N_{01} \equiv 5,\ N_{02} \equiv 10,\ N_{p1} \equiv 50 \quad (5.51)$$

FIGURE 5.12 The 2D Spectrum $G(k_1, k_2)$ of the 2D Discrete-Domain Gate Signal $g^{\circ}(n_1, n_2)$

(insert here)