CHAPTER 6 The Fourier Transform

6.1 THE MD FOURIER TRANSFORM OF CONTINUOUS DOMAIN SIGNALS

It has so far been established that continuous-time *periodic* MD signals $x(\mathbf{t})$ may be exactly represented by means of the MD Fourier Series and it has been shown that such signals possess an interpretation in the frequency domain in terms of MD phasors and MD harmonics. However, in many applications, MD signals are not periodic and may not be expressed as a Fourier Series. For example, all finite duration MD signals are essentially nonperiodic.

The Fourier Transform may be used to obtain the frequency domain characteristics of nonperiodic signals. In this section, the Fourier Transform of *continuous-*

domain MD signals is described. Subsequently, it is shown that the Fourier Transform of *discrete-domain* signals may be described in a similar way.

It will also be shown that the Fourier Transform can be used to determine the energy distribution of a MD signal in the frequency domain; a result that is fundamentally useful for the design of many MD signal processing algorithms, including the design of MD filters.

Consider an amplitude bounded continuous-domain MD signal $x(\mathbf{t})$, $\mathbf{t} \in \mathbf{R^m}$. The MD Fourier Transform $X(j\omega)$ of $x(\mathbf{t})$ is defined as follows

$$X(j\omega) \equiv \int_{t_m = -\infty}^{t_m = \infty} \dots \int_{t_3 = -\infty}^{t_m = \infty} \int_{t_2 = -\infty}^{t_m = \infty} \int_{t_1 = -\infty}^{t_m = \infty} x(\mathbf{t}) exp[-j\omega^{\mathrm{T}}\mathbf{t}] dt_1 dt_2 dt_3 \dots dt_m \quad (6.1)$$

Writing

$$\int_{\mathbf{t} = -\infty}^{\mathbf{t} = \infty} = \int_{t_m = -\infty}^{t_m = \infty} \dots \int_{t_3 = -\infty}^{t_m = \infty} \int_{t_2 = -\infty}^{t_m = \infty} \int_{t_1 = -\infty}^{t_m = \infty} \quad (6.2)$$

and

$$d\mathbf{t} = dt_1 dt_2 dt_3 \dots dt_i \dots dt_m \quad (6.3)$$

we may write equation (6.1) in the compact form

$$X(j\omega) \equiv \int_{\mathbf{t} = -\infty}^{\infty} x(\mathbf{t}) exp(-j\omega^{\mathrm{T}}\mathbf{t}) d\mathbf{t} \quad (6.4)$$

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The Inverse MD Fourier Transform is given by

$$x(\mathbf{t}) = \frac{1}{(2\pi)^m} \int_{\omega = -\infty}^{\infty} X(j\omega) exp(\omega^{\mathrm{T}}\mathbf{t}) d\omega \quad (6.5)$$

where

$$\int_{\omega=-\infty}^{\infty} \equiv \int_{\omega_m=-\infty}^{\infty} \cdots \int_{\omega_3=-\infty}^{\infty} \int_{\omega_2=-\infty}^{\infty} \int_{\omega_1=-\infty}^{\infty} \quad (6.6)$$

and

$$d\omega = d\omega_1 d\omega_2 d\omega_3 \ldots d\omega_i \ldots d\omega_m \quad (6.7)$$

The MD Fourier Transform operation on $x(\mathbf{t})$ in equation (6.4) is written, for brevity, as

$$\mathbf{F}^{\mathbf{m}} \quad (6.8)$$

so that

$$X(j\omega) = \mathbf{F}^{\mathbf{m}}[x(\mathbf{t})] \quad (6.9)$$

and, similarly, the MD Inverse Fourier Transform operation on $X(j\omega)$ in equation (6.5) is written

$$\mathbf{F}^{-\mathbf{m}}[\ ] \quad (6.10)$$

so that

$$x(\mathbf{t}) = \mathbf{F}^{-\mathbf{m}}[X(j\omega)] \quad (6.11)$$

We shall write the MD transform pair as

$$x(\mathbf{t}) \Leftrightarrow X(j\omega) \quad (6.12)$$

It should be noted that the regions of integration for $\mathbf{t}$ and ω in equations (6.3) and (6.4) are the entire MD region $\mathbf{R}^{\mathbf{m}}$.

The MD integral in equation (6.4) must exist if $X(j\omega)$ is to exist. A sufficient condition for this integral to exist is that $x(\mathbf{t})$ be square integrable over $\mathbf{R}^{\mathbf{m}}$, according to the definition of square integrability in equation (???) ; that is, $x(\mathbf{t})$ is a finite energy signal. Periodic signals have infinite energy and therefore their Fourier Transform does not exist. Infinite duration signals have a Fourier Transform if they are finite energy signals.

The function $|X(j\omega)|$ is referred to as the magnitude spectrum and the function $arg\ X(j\omega)$ as the phase spectrum of the signal $x(\mathbf{t})$. The magnitude and phase spectra possess certain symmetries in the frequency domain $dmn\ \omega$ that are described in Section (6.4) under the discrete domain case.

6.1.1 Parseval's Theorem for Continuous-Domain MD Signals

Parseval's Theorem is important because it provides a relationship between the signal $x(\mathbf{t})$ in the continuous domain and the distribution of the energy of the signal in the frequency domain.

Consider a pair of generally complex MD signals $x(\mathbf{t})$ and $y(\mathbf{t})$. The energy of this signal pair at $\mathbf{t} = \infty$ is obtained directly from equations (???) and (???) and is given by

$$E(\infty) = \int_{\mathbf{t}=-\infty}^{\infty} x(\mathbf{t})y^*(\mathbf{t})d\mathbf{t} \quad (6.13)$$

For a complex MD signal pair, it may be shown from the definition of the MD Fourier transform in equation (6.3) that

$$E(\infty) = \int_{\mathbf{t}=-\infty}^{\infty} x(\mathbf{t})y^*(\mathbf{t})d\mathbf{t} = \frac{1}{(2\pi)^m}\int_{\omega=-\infty}^{\infty} \mathbf{X}(j\omega)\mathbf{Y}^*(j\omega)d\omega \quad (6.14)$$

The above expression is the MD version of **Parseval's Theorem** for continuous-domain signals. For a **single** complex MD signal, equation (6.14) reduces to

$$E(\infty) = \int_{\mathbf{t}=-\infty}^{\infty} |x(\mathbf{t})|^2 d\mathbf{t} = \frac{1}{(2\pi)^m}\int_{\omega=-\infty}^{\infty} |\mathbf{X}(j\omega)|^2 d\omega \quad (6.15)$$

The *real* function

$$\Phi(\omega) \equiv |\mathbf{X}(j\omega)|^2 / (2\pi)^m \quad (6.16)$$

is defined as the **MD energy density spectrum of** $x(\mathbf{t})$. It describes the disribution of the energy of $x(\mathbf{t})$ over the frequency domain ω.

6.1.2 Geometric Interpretation of Parseval's Theorem for the 2D and 3D Cases:

Consider, for example, a continuous-domain square-integrable 2D signal $x(t_1, t_2)$, as illustrated in Figure (6.1). The total energy of the signal $E(\infty)$, as shown in Figure (6.2)., is obtained by integrating $|x(t_1, t_2)|^2$ over the infinite-extent planar region $\mathbf{R}^2$. The total energy is, by definition, the volume under the surface $|x(t_1, t_2)|^2$ in Figure (6.2).

FIGURE 6.1 A Continuous-Domain Square-Integrable Signal $x(t_1, t_2)$

(insert here)

FIGURE 6.2 Total Energy as Integral Under the 2D Surface $x(t_1, t_2)$

(insert here)

The Fourier Transform of $x(t_1, t_2)$ is generally a complex function, as illustrated in Figure (6.3). The real energy density spectrum

$$\Phi(\omega_1, \omega_2) \equiv |\mathbf{X}(j\omega_1, j\omega_2)|^2 / (2\pi)^2 \qquad \textbf{(6.17)}$$

may be obtained as illustrated in Figure (6.4). The total energy of the signal may be interpreted, according to the right side of equation (6.15), as the volume under the surface $\Phi(\omega_1, \omega_2)$ in Figure (6.2). The function $\Phi(\omega_1, \omega_2)$ is therefore appropriately referred to as the energy density function because it describes the distribution of signal energy throughout the 2D frequency plane. *It is often possible to enhance 2D signals by altering their energy density function* $\Phi(\omega_1, \omega_2)$ *in a specific and useful way*. We will return to this topic.

FIGURE 6.3 The 2D Fourier Transform

FIGURE 6.4 The 2D Energy Density Spectrum $\Phi(\omega_1, \omega_2)$

The 3D case is an obvious extension of the above 2D example. The total energy $E(\infty)$ is then interpreted as the integral of $|x(t_1, t_2, t_3)|^2$ over the entire 3D region $\mathbf{R}^3$ and the energy density spectrum $\Phi(\omega_1, \omega_2, \omega_3)$ is the frequency distribution of the total energy $E(\infty)$ over the frequency volume $\mathbf{R}^3$. The value of the integral of $\Phi(\omega_1, \omega_2, \omega_3)$ over some finite volume $\Omega \subset \mathbf{R}^3$ of the frequency space is the energy of the signal contained in that region of the 3D frequency space.

Example 19 Example A 1D Continuous-Domain Gate Pulse $\mathbf{g(t_1)}$

Consider the 1D gate pulse $g(t_1)$ shown in Figure (6.5). This is a nonperiodic finite-energy signal, having amplitude A over the interval $-T_0 \le t_1 \le T_0$ and zero amplitude outside of this interval.

FIGURE 6.5 A 1D Continuous-Domain Gate Signal $g(t_1)$

Before finding the Fourier Transform $G(j\omega_1)$ and the energy density spectrum $\Phi(\omega_1)$, we note that the total energy $E(\infty)$ is easily obtained by means of the integral in equation (6.15) and is therefore given by

$$E(\infty) = \int_{t_1 = -\infty}^{\infty} |g(t_1)|^2 dt_1 = \int_{t_1 = -\infty}^{\infty} A^2 dt_1 = 2A^2 T_0 \qquad \textbf{(6.18)}$$

as illustrated in Figure (6.5) . The Fourier Transform of $g(t_1)$ is given by

$$G(j\omega_1) = \int_{t_1 = \infty}^{\infty} g(t_1) exp(-j\omega_1 t_1) dt_1 = \int_{t_1 = -T_0}^{T_0} A exp(-j\omega_1 t_1) dt_1 \qquad \textbf{(6.19)}$$

The above integral may be evaluated in a similar way to the calculation in Example (25), leading directly to

$$G(j\omega) = 2AT_0 \operatorname{sinc}(\omega_1 T_0) \qquad \textbf{(6.20)}$$

so that the energy density spectrum is given by

$$\Phi(\omega_1) = \frac{|G(j\omega_1)|^2}{2\pi} = \frac{2A^2 T_0^2 [\operatorname{sinc}(\omega_1 T_0)]^2}{\pi} \qquad \textbf{(6.21)}$$

The Fourier Transform $G(j\omega_1)$ and the energy density spectrum $\Phi(\omega_1)$ are shown in Figure (6.6). The total energy $E(\infty)$ is the area under the curve $\Phi(\omega_1)$ and the shaded area corresponds to the energy in $g(t_1)$ that is in the frequency intervals $\omega_{11} \le \omega_1 \le \omega_{12}$ *and* $-\omega_{12} \le \omega_1 \le -\omega_{11}$. It should be noted that $G(j\omega_1)$ and $\Phi(\omega_1)$ are *continuous functions of frequency* ω_1 . This is characteristic of finite energy signals.

FIGURE 6.6 The 2D Energy Density Spectrum

It will be observed that the shape of $\Phi(\omega_1)$ depends on the duration $2T_0$ of the pulse $g(t_1)$ in a characteristic way; if the duration of the pulse is reduced, by reducing $2T_0$, then the total energy is also reduced in proportion to $2T_0$, according to equation (6.18). The zero crossings of $\Phi(\omega_1)$ are at integer multiples of π / T_0 and it can be observed that $\Phi(\omega_1)$ therefore undergoes a proportionate compressional scaling on the frequency axis as T_0 is reduced. The amplitude of $\Phi(\omega_1)$ is reduced as the square of T_0. This behaviour is characteristic of the energy distribution of signals; as the duration is reduced, the bandwidth (or band of frequencies) occupied by the energy density spectrum increases.

Finally, Parseval's Theorem may be confirmed for this example by substituting the following integral equation

$$\int_{-\infty}^{\infty} |\sin(x)|^2 dx = \pi \tag{6.22}$$

which is available in published tables of integrals, into equation (6.21), leading directly to

$$E(\infty) = \frac{1}{2\pi} \int_{t=-\infty}^{\infty} 4A^2 T_0^2 (\sin(\omega_1 T_0))^2 d\omega_1 = 2A^2 T_0 \tag{6.23}$$

which corresponds to the calculation of energy obtained in the t domain (6.18).

Example 20 Example A 2D Continuous Domain Gate Pulse $g(t_1, t_2)$

Consider the 2D continuous domain gate pulse shown in Figure (6.7), having amplitude A in the interval $(-T_{01} \le t_1 \le T_{01}, -T_{02} \le t_2 \le T_{02})$ and zero amplitude elsewhere.

FIGURE 6.7 A 2D Continuous-Domain Gate Signal

The total energy is given, according to equations (2.116) and (2.125), by

$$E(\infty) = \int_{t_2=-\infty}^{\infty} \int_{t_1=-\infty}^{\infty} |g(t_1, t_2)|^2 dt_1 dt_2 = \int_{t_2=-T_{02}}^{T_{02}} \int_{t_1=-T_{01}}^{T_{01}} A^2 dt_1 dt_2 = 4A^2 T_{01} T_{02} \quad \textbf{(6.24)}$$

The 2D Fourier Transform is obtained from equation (6.3) as

$$G(j\omega_1, j\omega_2) = A \int_{t_2=-T_{02}}^{T_{02}} \int_{t_1=-T_{01}}^{T_{01}} exp[-j(\omega_1 t_1 + \omega_2 t_2)] dt_1 dt_2$$

$$= A \int_{t_1=-T_{01}}^{T_{01}} exp(-j\omega_1 t_1) dt_1 \int_{t_2=-T_{02}}^{T_{02}} exp(-j\omega_2 t_2) dt_2 \quad \textbf{(6.25)}$$

$$= a[2 T_{01} sinc(\omega_1 T_{01})][2T_{02} sinc(\omega_2 T_{02})] \quad \textbf{(6.26)}$$

where the terms in the square brackets are obtained from the integrals in equation (6.25) using the method in Example (23). This function is shown in Figure (6.8).

FIGURE 6.8 The 2D Fourier Transform of the 2D Continuous-Domain Gate Signal $G(j\omega_1, j\omega_2)$

The 2D energy density spectrum $\Phi(\omega_1, \omega_2)$ is given by

$$\Phi(\omega_1, \omega_2) = \frac{|G(j\omega_1, j\omega_2)|^2}{(2\pi)^2} = \frac{16A^2T_{01}^2T_{02}^2[sinc(\omega_1 T_{01})]^2[sinc(\omega_2 T_{02})]^2}{(2\pi)^2} \quad \textbf{(6.27)}$$

This expression is proportional to the product of two $sinc^2[\ ]$ functions and is shown in Figure (6.9). Clearly, the effect of reducing the duration T_{02} of the gate signal in the direction t_2, for example, is to increase the bandwidth of $\Phi(\omega_1, \omega_2)$ *in the direction* ω_2. It is straightforward to calculate $E(\infty)$ from $\Phi(\omega_1, \omega_2)$ by double integration over the frequency domain so that

$$E(\infty) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{16A^2T_{01}^2T_{02}^2[\mathrm{sin}c(\omega_1 T_{01})]^2[\mathrm{sin}c(\omega_2 T_{02})]^2}{(2\pi)^2}\right)d\omega_1 d\omega_2 \quad \textbf{(6.28)}$$

which, using equation ( ) simplifies to

$$E(\infty) = 4A^2T_{01}T_{02} \quad \textbf{(6.29)}$$

and corresponds to the t domain calculation in ( ).

FIGURE 6.9 The 2D Energy Density Spectrum

6.1.3 Linear Transformation Properties of the MD Fourier Transform

It is often the case that one knows the MD transform of a particular signal $x(\mathbf{t})$ and is interested to know the transform of a related signal $x(\mathbf{u})$ where $\mathbf{u}$ is some new coordinate MD system that is related to the coordinate system t by a linear transformation of the form

$$\mathbf{u} = \mathbf{At} \quad \textbf{(6.30)}$$

The transformation **A** might represent a combination of scalings and rotations of the MD signal. The above scaling matrix G and the more general rotational matrix $\mathbf{R_d}$ in Chapter 2 are specific examples of such transformations. We want to establish the effect of such transformations in the domain of the Fourier Transform.

Mathematical Preliminaries:

We shall need to call on the elementary property of matrices that

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{6.31}$$

and

$$(\mathbf{A}^T)^T = \mathbf{A} \tag{6.32}$$

and, for a nonsingular matrix (that is, where $\mathbf{A}^{-1}$ exists)

$$\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \tag{6.33}$$

Finally, we shall write the transpose of the inverse $\mathbf{A}^{-1}$ as $\mathbf{A}^{-T}$.

We shall also need the following fundamental result from elementary calculus, involving multiple integrals under a change of variables from $t_1, t_2, \ldots, t_m$ to $u_1, u_2, \ldots, u_m$. It may be shown [ref Sneddon p394] that

$$\int_{\mathbf{T}}\cdots\int x(t_1, t_2, \ldots, t_m)dt_1dt_2\ldots dt_m = \int_{\mathbf{U}}\cdots\int x(t_1, t_2, \ldots, t_m)|\mathbf{J_{tu}}|du_1du_2\ldots du_m \tag{6.34}$$

where $\mathbf{J_{tu}}$ is the Jacobian determinant

$$\mathbf{J_{tu}} = \begin{vmatrix} \frac{\partial t_1}{\partial u_1} & \frac{\partial t_1}{\partial u_2} & \cdots & \frac{\partial t_1}{\partial u_m} \\ \frac{\partial t_2}{\partial u_1} & \frac{\partial t_2}{\partial u_2} & \cdots & \frac{\partial t_2}{\partial u_m} \\ : & : & : & : \\ \frac{\partial t_m}{\partial u_1} & \frac{\partial t_m}{\partial u_2} & \cdots & \frac{\partial t_m}{\partial u_m} \end{vmatrix} \tag{6.35}$$

and where the transformation of variables $\mathbf{t} \to \mathbf{u}$ maps the MD region of integration **T** into the corresponding MD region **U**. The subscipt $_{\mathbf{tu}}$ on the Jacobian indicates that the transformation or mapping is from **t** to **u**. It is easily shown, by direct partial differentiation of the elements of **A** according to equation (6.35) that the Jacobian determinant $\mathbf{J_{tu}}$ for the transformation in equation (6.30) is simply given by

$$\mathbf{J_{tu}} = \det \mathbf{A} \tag{6.36}$$

Consider now the inverse transformation

$$\mathbf{t} = \mathbf{A}^{-1}\mathbf{u} \tag{6.37}$$

The Jacobian determinant $\mathbf{J_{tu}}$ is therefore given by

$$\mathbf{J_{ut}} = \mathbf{det}\ \mathbf{A}^{-1} \tag{6.38}$$

which, according to equation (6.33), is equivalent to

$$\mathbf{J_{ut}} = \frac{1}{\det \mathbf{A}} \tag{6.39}$$

Proof of the Tran(6.32)ormation Property:

Given $x(\mathbf{t})$ and that $\mathbf{A}$ is nonsingular, we want to derive the MD Fourier Transform of $x(\mathbf{At})$ which we shall write as $X'(j\omega)$. By definition,

$$X'(j\omega) = \int_{\mathbf{t}=-\infty}^{\infty} x(\mathbf{At})exp(-j\omega^{\mathrm{T}}\mathbf{t})d\mathbf{t} \tag{6.40}$$

Assuming that $\mathbf{A}$ is nonsingular, substituting equations (6.30) and (6.37) into (6.40) gives

$$X'(j\omega) = \int_{\mathbf{t}=-\infty}^{\infty} x(\mathbf{u})exp(-j\omega^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u})d\mathbf{t} \tag{6.41}$$

Using the property of matrix transposes from equations (6.31) and (6.32) gives

$$\omega^{\mathrm{T}}\mathbf{A}^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}(\omega^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}^{-\mathrm{T}}\omega \tag{6.42}$$

Substituting equation (6.42) in (6.41) gives

$$X'(j\omega) = \int_{\mathbf{t}=-\infty}^{\infty} x(\mathbf{u})exp(-j\omega^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u})d\mathbf{t} \tag{6.43}$$

Using the Jacobian determinant from equation (6.36) and using equation (6.34) to change the variables and limits of integration to $\mathbf{u}$ gives

$$X'(j\omega) = \int_{\mathbf{u}=-\infty}^{\infty} x(\mathbf{u}) exp(-j\mathbf{A}^{-T}\omega\mathbf{u})\left(\frac{1}{|\det \mathbf{A}|}\right) d\mathbf{u} \quad \textbf{(6.44)}$$

We have almost arrived at the required result. If we define

$$\Omega \equiv \mathbf{A}^{-T}\omega \quad \textbf{(6.45)}$$

then equation (6.44) becomes

$$X'(j\omega) = \frac{1}{|\det \mathbf{A}|}\int_{\mathbf{u}=-\infty}^{\infty} x(\mathbf{u}) exp(-j\Omega\mathbf{u}) d\mathbf{u} \quad \textbf{(6.46)}$$

which is the required result. Inspection of equation (6.46) reveals that this is the MD Fourier Transform of $x(\mathbf{t})$ *except that* Ω *has been replaced by the new frequency variable* $\Omega \equiv \mathbf{A}^{-T}\omega$ and the transform has been multiplied by $\frac{1}{|\det \mathbf{A}|}$. We have, in summary,

$$\text{if } x(\mathbf{t}) \Leftrightarrow X(j\omega)$$

$$\text{then } x(\mathbf{A}\mathbf{t}) \Leftrightarrow \frac{X(j\mathbf{A}^{-T}\omega)}{|\det \mathbf{A}|} \quad \textbf{(6.47)}$$

which we refer to as the **Linear Transformation Property** of the MD Fourier Transform.

6.1.4 MD ROTATION, SCALING AND SHIFT OPERATIONS

The Linear Transformation Property is now used to derive a number of useful properties that relate operations in the domain $\mathbf{t}$ of $x(\mathbf{t})$ to operations in the frequency domain ω of $X(j\omega)$.

MD Rotations

Assume a linear transformation $\mathbf{R}$ where the matrix $\mathbf{R}$ is defined to be real and orthogonal. Then, from the Linear Transformation property in equation ( )

$$x(\mathbf{R}\mathbf{t}) \overset{md}{\Leftrightarrow} X(j\mathbf{R}^{-T}\omega)/|\det \mathbf{R}| \quad \textbf{(6.48)}$$

where, from Section 2.?.?, R has the properties that

$$\mathbf{R}^{-T} = \mathbf{R} \text{ and} \det \mathbf{R} = \pm 1 \quad \textbf{(6.49)}$$

Substituting equations ( ) into ( ) gives the following transform pair under MD rotation

$$x(\mathbf{R}\mathbf{t}) \overset{\text{md}}{\Leftrightarrow} X(j\mathbf{R}^{-T}\omega) \tag{6.50}$$

We conclude that the rotation of a signal $x(\mathbf{t})$ *in the* $\mathbf{t}$ *domain causes an identical rotation of* $X(j\omega)$ *in the* ω *domain*. The transformation $\mathbf{R}$ is illustrated in Figure for the 2D case $\mathbf{R} = \mathbf{R_1}\mathbf{R_2}$.

FIGURE 6.10 The 2D Rotation

MD Scaling

Let the dimensions of the MD signal be scaled using the linear transformation diagonal matrix

$$\Gamma \equiv diag[\tau_1, \tau_2, \ldots, \tau_m] = \begin{bmatrix} \tau_1 & 0 & . & . & 0 \\ 0 & \tau_2 & & & 0 \\ . & & . & & . \\ . & & & . & . \\ 0 & & & . & . \\ 0 & 0 & . & . \; 0 & \tau_m \end{bmatrix} \tag{6.51}$$

corresponding to a scaling of the orthogonal axes according to

$$x(\Gamma\mathbf{t}) = x(\tau_1 t_1, \tau_2 t_2, \ldots, \tau_m t_m) \tag{6.52}$$

Now, it follows from equation ( ) that

$$\Gamma^{-1} = diag[1/\tau_1, 1/\tau_2, \ldots, 1/\tau_i, \ldots, 1/\tau_m] \tag{6.53}$$

and

$$\det \Gamma = \tau_1 \tau_2 \ldots \tau_i \ldots \tau_m \tag{6.54}$$

Substituting equations ( ) and ( ) in equation ( ) gives the following result

$$x(\Gamma \mathbf{t}) \overset{\text{md}}{\Leftrightarrow} X(j\Gamma^{-1}\omega)/(\tau_1 \tau_2 \ldots \tau_i \ldots \tau_m) \tag{6.55}$$

or, equivalently,

$$x(\tau_1 t_1, \ldots \tau_i t_i, \ldots \tau_m t_m) \overset{\text{md}}{\Leftrightarrow} X(j\omega_1/\tau_1, \ldots j\omega_i/\tau_i, \ldots j\omega_m/\tau_m)/(|\tau_1 \tau_2 \ldots \tau_i \ldots \tau_m|) \tag{6.56}$$

Scaling the signal $x(t)$ on the dimensional axes in the domain t is equivalent to reciprocal scaling of the dimensional axes of the Fourier Transform in the ω domain and dividing the amplitude of the Fourier Transform by the magnitude of the product of the scaling terms $|\tau_1 \tau_2 \ldots \tau_i \ldots \tau_m|$.

Combinations of Scalings and Rotations:

It is easily shown that for any two nonsingular matrices

$$(\mathbf{R_2}\Gamma)^{-T} = \mathbf{R_2}^{-T}\Gamma^{-T} \tag{6.57}$$

However, all scalings Γ satisfy

$$\Gamma^{-T} = \Gamma^{-1} \tag{6.58}$$

and all rotations are orthogonal matrices satisfying

$$\mathbf{R_2}^{-T} = \mathbf{R_2} \tag{6.59}$$

so that, from equations ( ), ( ) and ( ),

$$\mathbf{R_2}^{-T} = \mathbf{R_2}^{-1} \tag{6.60}$$

It follows directly from equations ( ), ( ) and ( ) that scaling-followed-by-rotation is characterized by the transform pair

$$x(\mathbf{R_2}) \overset{\text{md}}{\Leftrightarrow} X(j\mathbf{R_2}^{-1}\omega)/|\det \Gamma| \tag{6.61}$$

Example 21 Scaling and Rotation of a 2D Signal

Consider a 2D signal 2d $x(t_1, t_2)$ as shown in Figure (6.11), having the transform pair $x(t) \Leftrightarrow X(j\omega)$ and where $|X(j\omega)|$ is shown in Figure (6.11).

FIGURE 6.11 2D Example of Scaling and Rotation

Now, let $x(t_1, t_2)$ be compressed by the the factor 2 on the t_1 axis and stretched by the factor 3 on the t_2 axis as shown in Figure (6.12), followed by a 45 degree anticlockwise rotation, as shown in Figure (6.12). The corresponding transformations in the domain t are given by

$$\Gamma \equiv diag[1/2, 3] = \begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix} \quad (6.62)$$

and

$$R_2 = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin\pi/4 & \cos(\pi/4) \end{bmatrix} \quad (6.63)$$

We note that

$$\det(\Gamma) = (2)(1/3) = 2/3 \quad (6.64)$$

and

$$R_2^{-T} = R_2^{-1} = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin\pi/4 & \cos(\pi/4) \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix} \quad (6.65)$$

FIGURE 6.12 2D Compression and Scaling

A simple geometrical interpretation is now possible. From the above expression and equation ( ), the Fourier Transform $X(jR_2^{-1}t)/|det\ \Gamma|$ is obtained from $X(j\omega)$ by scaling the frequency axis ω_1 by the factor 2, the frequency axis ω_2 by the factor $1/3$ and scaling the amplitude by $3/2$ as shown in Figure . Finally, the Fourier Transform $X(jR_2^{-1}t)/|det\ \Gamma|$ is obtained by rotating the function in Figure by 45 degrees anticlockwise, as shown in Figure .

FIGURE 6.13 Effects of Compression-Scaling and Rotation in the 2D Frequency Domain

MD Shift Operations

Consider that the MD signal $x(t)$ is shifted in the domain t by the vector b so that the resultant shifted signal is $x(t-b)$. Then,

$$\mathbf{F^m}[x(\mathbf{t}-\mathbf{b})] = \int_{t=\infty}^{\infty} x(\mathbf{t}-\mathbf{b})exp[-j\omega^{\mathrm{T}}\mathbf{t}]d\mathbf{t} \quad \textbf{(6.66)}$$

Substituting $u \equiv t-b$ and $du \equiv dt$ in the right side of the above equation and changing the variable and limits of integration gives

$$\mathbf{F}^{\mathbf{m}}[x(\mathbf{t}-\mathbf{b})] = \int_{\mathbf{u} = \infty}^{\infty} x(\mathbf{u})\, exp[-j\omega^{\mathrm{T}}(\mathbf{u}+\mathbf{b})]d\mathbf{u}$$

$$= exp[-j\omega^{T}b] \int_{\mathbf{u} = \infty}^{\infty} x(\mathbf{u})exp[-j\omega^{\mathrm{T}}\mathbf{u}]d\mathbf{u} = exp[-j\omega^{T}b]\mathbf{F}^{\mathbf{m}}[x(\mathbf{t})] \quad \textbf{(6.67)}$$

so that

$$\mathbf{F}^{\mathbf{m}}[x(\mathbf{t}-\mathbf{b})] = exp[-j\omega^{\mathrm{T}}\mathbf{b}]X(j\omega) \quad \textbf{(6.68)}$$

This is the MD Shift Property. It shows that the effect in the frequency domain of shifting a signal by the vector $\mathbf{b}$ is to multiply the MD Fourier Transform by the constant exponential term $exp[-j\omega^{1}\mathbf{b}]$; this leaves the magnitude of the Fourier Transform unaltered while subtracting $\omega^{1}\mathbf{b}$ from its phase.

Example 22 The 2D Shift Operation

Consider that the 2D signal $x(\mathbf{R_2 t})$ is shifted by the vector

$$\mathbf{b} \equiv [b_1 \; b_2]^{1} \quad \textbf{(6.69)}$$

as shown in Figure 3.15(a), corresponding to the shifted signal $x(\mathbf{R}_2\mathbf{t}-\mathbf{b})$. Then, according to equation ( ), the magnitude spectrum of this shifted signal is the same as Figure $3.14(f)$ and is essentially unaltered by the shift operation. However, the 2D phase spectrum of Figure 3.15(a) is altered by the shift; the phase correction due to shifting is given by

$$-\omega^{1}b = -(\omega_1 b_1 + \omega_2 b_2) \quad \textbf{(6.70)}$$

which is the planar function shown in Figure $3.15(c).

FIGURE 6.14 The 2D Shift Operation

6.2 THE FOURIER TRANSFORM OF DISCRETE DOMAIN SIGNALS

The Fourier transform of discrete domain signals $x(\mathbf{n})$ is an important special case of the above continuous domain analysis, primarily because signal are most often processed as sampled versions of the continuous domain signal. The constraints that are imposed by the sampling process will be considered in the next chapter. For now, we assume the existence of an amplitude-bounded finite-energy uniformly-sampled discrete domain MD signal $x(n_1T_1, n_2T_2, \dots, n_mT_m)$. The Fourier transform of such a signal is given by

$$X(j\omega) = \sum_{n=-\infty}^{n=\infty} x(n_1T_1, n_2T_2, \dots, n_mT_m) exp(-j(\omega_1 n_1 T_1 + \omega_2 n_2 T_2 + \dots + \omega_m n_m T_m)) \tag{6.71}$$

FOURIER TRANSFORM OF A DISCRETE-DOMAIN SIGNAL

which is usually written in the distance normalized form by assuming that T_i is unity for $i = 1, 2, \dots, m$, giving

$$X(j\omega) = \sum_{\mathbf{n}=-\infty}^{\mathbf{n}=\infty} x(\mathbf{n}) exp(-j\omega^T \mathbf{n}) \tag{6.72}$$

FOURIER TRANSFORM OF A DISCRETE DOMAIN SIGNAL HAVING UNIT DISTANCE BETWEEN SAMPLES

where $n = [n_1\ n_2\ \dots\ n_m]^1$. The corresponding inverse Fourier transform is given by

$$x(\mathbf{n}) = \frac{1}{(2\pi)^m} \int_{\omega = -\pi}^{\omega = \pi} X(j\omega) exp[j\omega^T \mathbf{n}] \tag{6.73}$$

INVERSE FOURIER TRANSFORM OF A DISCRETE DOMAIN SIGNAL

The above transform pairs are written

$$x(\mathbf{n}) \overset{md}{\Leftrightarrow} X(j\omega) \tag{6.74}$$

A sufficient condition for the existence of $X(j\omega)$ is that $x(\mathbf{n})$ be square summable and therefore satisfy equation ( ). Equivalently, all finite energy discrete domain signals have a Fourier transform.

6.2.1 Parseval's Theorem

The relationship between the energy of (generally complex) discrete domain signals in the $dmn\ n$ and the domain $dmn\ \omega$ is the discrete version of Parseval's Theorem (equation ( )) and is given by

$$E(\infty) \equiv \sum_{n = -\infty}^{n = \infty} x(\mathbf{n}) y^*(\mathbf{n}) = \frac{1}{(2\pi)^m} \int_{\omega = -\pi}^{\omega = \pi} X(j\omega) Y^*(j\omega) d\omega \tag{6.75}$$

PARSEVAL'S THEOREM IN THE DISCRETE DOMAIN

This relationship between the energy of a two-pair in $dmn\ \mathbf{t}$ and the Fourier transforms of the two-pair signals in $dmn\ \omega$ is important because it is the basis for the design of the widely employed class of passive high performance discrete domain filter algorithms. Some of these algorithms are discussed in Chapter ? . With $y(\mathbf{n})$ set equal to $x(\mathbf{n})$, the above result gives the energy of a single signal $x(\mathbf{n})$ as

$$E(\infty) \equiv \sum_{n = -\infty}^{n = \infty} |x(\mathbf{n})|^2 = \frac{1}{(2\pi)^m} \int_{\omega = -\pi}^{\omega = \pi} |X(j\omega)|^2 d\omega \tag{6.76}$$

and equation ( ) is also the MD energy density function of a discrete domain signal.

6.3 PROPERTIES OF THE MD FOURIER TRANSFORM

Many widely encountered operations on a signal $x(\mathbf{t})$ in *dmn* $\mathbf{t}$ correspond to straightforward equivalent operations on the Fourier transform $X(j\omega)$ in *dmn* ω. Some of the more important of the operations are given below for the continuous domain transform pair in equation ( ). Similar relations hold for the continuous domain case, except where specifically noted.

Linearity Property Given

$$x_1(\mathbf{t}) \overset{md}{\Leftrightarrow} X_1(j\omega) \quad \text{and} \quad x_2(\mathbf{t}) \overset{md}{\Leftrightarrow} X_2(j\omega) \tag{6.77}$$

then, for any complex numbers α and β,

$$\alpha x_1(\mathbf{t}) + \beta x_2(\mathbf{t}) \overset{md}{\Leftrightarrow} \alpha X_1(j\omega) + \beta X_2(j\omega) \tag{6.78}$$

This property allows the Fourier transform to be widely employed. For example, with $\alpha \equiv \beta \equiv 1$, equation ( ) simply implies that the transform of the sum (that is superposition) of a set of signals may be obtained from the sum of their individual Fourier transforms. This is the Principle of Superposition and is fundamentally important and widely employed in the analysis and design of signal processing systems.

Complex Conjugation Property

It follows directly from equation ( ) that

$$x^*(\mathbf{t}) \overset{md}{\Leftrightarrow} X^*(-j\omega) \tag{6.79}$$

Real Part Properties

It follows from equation ( ) that

$$\mathbf{Re}[x(\mathbf{t})] \overset{md}{\Leftrightarrow} \frac{1}{2}[X(j\omega) + X^*(-j\omega)] \tag{6.80}$$

and

$$\mathbf{Re}[X(j\omega)] \overset{md}{\Leftrightarrow} \frac{1}{2}[x(\mathbf{t}) + x^*(-\mathbf{t})] \tag{6.81}$$

Imaginary Part Properties

It follows from equation ( ) that

$$j\mathbf{Im}[x(\mathbf{t})] \overset{md}{\Leftrightarrow} \frac{1}{2}[X(j\omega) - X^*(-j\omega)] \quad (6.82)$$

and

$$j\mathbf{Im}[X(j\omega)] \overset{md}{\Leftrightarrow} \frac{1}{2}[x(\mathbf{t}) - x^*(-\mathbf{t})] \quad (6.83)$$

Frequency Domain Properties for Real Signals $x(\mathbf{t})$

Given that the signal $x(\mathbf{t}) \in \mathbf{R}^m$, then it follows from equation ( ) that

$$X(j\omega) = X^*(-j\omega) \quad (6.84)$$

from which it follows directly that

$$\mathbf{Re}[X(j\omega)] = \mathbf{Re}[X^*(-j\omega)] \quad (6.85)$$

and

$$\mathbf{Im}[X(j\omega)] = -\mathbf{Im}[X^*(-j\omega)] \quad (6.86)$$

Equation ( ) is the *reflective conjugate symmetry* property of real MD signals; symmetry properties are pursued in more detail in Section 3.??. A number of additional properties follow, for real signals $x(\mathbf{t})$, as a consequence of the reflective conjugate symmetry property (see Problem 3.??). It is easily shown that

$$|X(j\omega)| = |X^*(-j\omega)| \quad (6.87)$$

and

$$\mathbf{Re}[X(j\omega)] = \mathbf{Re}[X(-j\omega)] \quad (6.88)$$

implying that the magnitude Fourier spectrum $|X(j\omega)|$ and the real part Fourier spectrum $\mathbf{Re}[X(j\omega)]$ of real signals have even symmetry about the frequency domain origin $\omega = 0$ in all m frequency variables ω_i, $i = 1, 2, \ldots, m$. We shall later define this type of MD symmetry as centro-symmetry. Similarly,

$$\mathbf{Im}[X(j\omega)] = -\mathbf{Im}[X(-j\omega)] \quad (6.89)$$

implying that the imaginary part Fourier spectrum $\mathbf{Im}[X(j\omega)]$ has odd symmetry in all m frequency variables.

6.4 SYMMETRIES UNDER TRANSFORMATIONS OF THE SIGNAL $x(\mathbf{n})$ AND ITS FOURIER TRANSFORM

In many practical applications, the signal $x(\mathbf{t})$ is constrained by the nature of the problem at hand. In some applications, the signal may exhibit particular symmetries in $dmn\ \mathbf{t}$ that lead to corresponding symmetries in the frequency domain $dmn\ \omega$. There are many different kinds of symmetry constraints that can be analyzed and that have been reported in the literature [Reddy, Rajan, Swamy]. In general, a signal $x(\mathbf{n})$ is said to possess a **symmetry**, or **identity symmetry**, under the operation $\Phi[.]$ and over the domain $dmn\ \mathbf{t}$ if

$$\Phi[x(\mathbf{t})] = x(\mathbf{t}) \tag{6.90}$$

If

$$\Phi[x(\mathbf{t})] = -x(\mathbf{t}) \tag{6.91}$$

The signal $x(\mathbf{t})$ is said to be **antisymmetric** under the operation $\Phi[.]$. Similarly, if

$$\Phi[x(\mathbf{t})] = x^*(\mathbf{t}) \tag{6.92}$$

then $x(\mathbf{t})$ is said to be **conjugate symmetric** under the operation $\Phi[.]$ and if

$$\Phi[x(\mathbf{t})] = -x^*(\mathbf{t}) \tag{6.93}$$

then $x(\mathbf{t})$ is said to be **conjugate antisymmetric** under the operation $\Phi[.]$ The above symmetries become reflective symmetries if they are valid when $\mathbf{t}$ is replaced by $-\mathbf{t}$ on the right side. For example, $X(j\omega)$ in equation ( ) possesses **reflective conjugate antisymmetry**!

We shall consider some of the elementary symmetries of $x(\mathbf{t})$ that have practical implications in signal processing.

6.4.1 SYMMETRIES UNDER ORTHOGONAL TRANSFORMATIONS

Symmetries under the orthogonal transformations of equation ( ) are of particular interest; that is, the class of symmetries defined by

$$x(\mathbf{A}\mathbf{t}) = x(\mathbf{t}) \tag{6.94}$$

$$\text{where, by orthogonality, } \mathbf{A} \equiv \mathbf{A}^{-\mathrm{T}} \tag{6.95}$$

Recall that the lengths of vectors are invariant under the orthogonal transformations of equation ( ) . Equation ( ) therefore corresponds to the class of symmetries *for* *which there exists a length preserving real transformation* **A** *that leaves the signal unaltered.* It follows directly from equations ( ) and ( ) that, for such transformations,

$$X(j\omega) \overset{\text{md}}{\Leftrightarrow} x(\mathbf{t}) = x(\mathbf{A}\mathbf{t}) \overset{\text{md}}{\Leftrightarrow} X(j\omega) \tag{6.96}$$

so that

$$X(j\omega) = X(j\mathbf{A}\omega) \tag{6.97}$$

That is, symmetry under an orthogonal transformation on $x(\mathbf{t})$ in its $dmn\ \mathbf{t}$ implies the identical symmetry under the same transformation on $X(j\omega)$ in its domain $dmn\ \omega$.

There are many such transformations, some of which have particularly straightforward interpretations in the 2D and 3D cases. Some of them are considered in the following and some are considered in the problems at the end of this chapter. We consider symmetries under rotations and reflections.

Example 23 2D Rotation by 180 Degrees

In the 2D case, geometrical rotation of $x(n_1, n_2)$ by 180 degrees in $dmn\ \mathbf{n}$ corresponds to $\theta_1 \equiv \pi$ in equation ( ). Then, the rotation matrix $\mathbf{R_1}(=\mathbf{A})$ is given by

$$\mathbf{A} = \mathbf{R_1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.98}$$

so that

$$x(t_1, t_2) = x(-t_1, -t_2) \tag{6.99}$$

and

$$X(j\omega_1, j\omega_2) = X(-j\omega_1, -j\omega_2) \tag{6.100}$$

Example 24 2D Rotation by 90 Degrees

In the 2D case, geometrical rotation of $x(t_1, t_2)$ by 90 degrees in $dmn\ \mathbf{n}$ corresponds to $\theta_1 \equiv \pi/2$ in equation ( ). Then, the *orthogonal* rotation matrix $\mathbf{R_1}(=\mathbf{A})$ is given by

$$\mathbf{A} = \mathbf{R_1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{6.101}$$

so that

$$x(t_1, t_2) = x(t_2, -t_1) \quad \textbf{(6.102)}$$

and

$$X(j\omega_1, j\omega_2) = X(j\omega_2, -j\omega_1) \quad \textbf{(6.103)}$$

6.4.2 SYMMETRIES UNDER REFLECTIONS

Symmetries under reflections in $dmn\ \mathbf{t}$ lead to particular symmetries in the corresponding Fourier transform. The following symmetries under reflections are further examples of orthogonal transformations *so that identical symmetries exist in both the* $dmn\ \mathbf{t}$ *and the* $dmn\ \omega$.

Reflection Symmetry of $x(\mathbf{n})$ About One Axis Suppose that the signal is *equal to its own reflection* about the i th axis or dimension. Equivalently, the (generally complex) signal $x(\mathbf{t})$ has the property that it remains unaltered after a reversal in sign of the i th dimensional variable t_i; then

$$x(t_1, t_2, \ldots, t_i, \ldots, t_m) = x(t_1, t_2, \ldots, -t_i, \ldots, t_m) \quad \textbf{(6.104)}$$

and it follows directly from equation ( ) and equation ( ) that

$$x(t_1, t_2, \ldots, t_i, \ldots, t_m) \overset{md}{\Leftrightarrow} X(j\omega_1, j\omega_2, \ldots, -j\omega_i, \ldots, j\omega_m) \quad \textbf{(6.105)}$$

Clearly, symmetry of $x(\mathbf{t})$ about the i th axis in $dmn\ \mathbf{t}$ corresponds to symmetry of its generally complex Fourier transform $X(j\omega)$ about the same axis. The same result can be obtained by substituting $\mathbf{A} = diag[1, 1, \ldots, -1, \ldots, 1, 1]$ in equation ( ) , where -1 is the i th entry. This type of symmetry is also referred to as **twofold symmetry** about the i th axis because clearly the signal is unaltered if geometrically folded, in the 2D case, about the corresponding axis.

Reflection Symmetry of $x(\mathbf{t})$ About All Axes (Centro-Symmetry) It follows, by extension of ( ) to all dimensions, that if $x(\mathbf{t})$ has reflection symmetry about $\mathbf{t}$ in all dimensions then

$$x(\mathbf{t}) = x(-\mathbf{t}) \quad \textbf{(6.106)}$$

and

$$X(j\omega) = X(-j\omega) \quad \textbf{(6.107)}$$

This corresponds to $\mathbf{A} = -\mathbf{I}$ in equation ( ). This type of symmetry is often referred to as **centro-symmetry** and we state that the signal has **M-fold symmetry** about the **M** axes in $\mathbf{R}^{\mathbf{m}}$. It may be noted that the previously considered case of 2D

180 degree rotational symmetry also corresponds, in this particular example, to 2D centro-symmetry.

Diagonal , Quadrantal and Octagonal Symmetries In the 2D case, the definition of reflection symmetry about the diagonal 45 degree line

$$t_1 = t_2 \tag{6.108}$$

is given by

$$x(t_1, t_2) \equiv x(t_2, t_1) \tag{6.109}$$

as shown in Figure $3.??(b)$. The corresponding symmetry relationship in the frequency domain is

$$X(j\omega_1, j\omega_2) = X(j\omega_2, j\omega_1) \tag{6.110}$$

The definition of reflection symmetry about the diagonal -45 degree line

$$t_1 = -t_2 \tag{6.111}$$

is given by

$$x(t_1, t_2) \equiv x(-t_2, -t_1) \tag{6.112}$$

as shown in Figure $3.??(c)$. The corresponding symmetry relationship in the frequency domain is

$$X(j\omega_1, j\omega_2) = X(-j\omega_2, -j\omega_1) \tag{6.113}$$

A 2D signal having symmetry about both the 45 and -45 degree lines satisfies both equations ( ) and ( ) and is said to have **2D diagonal symmetry**.

A 2D signal is said to be **Quadrantally Symmetric** if it has twofold symmetry about both axes so that

$$x(t_1, t_2) \equiv x(-t_1, t_2) \equiv x(t_1, -t_2) \equiv x(-t_1, -t_2) \tag{6.114}$$

as shown in Figure $3.??(d)$. Clearly, Quadrantal Symmetry implies centro-symmetry as well as symmetry in all four quadrants from which it follows that

$$X(j\omega_1, j\omega_2) \equiv X(-j\omega_1, j\omega_2) \equiv X(j\omega_1, -j\omega_2) \equiv X(-j\omega_1, -j\omega_2) \tag{6.115}$$

A 2D signal is defined to have **Octagonal Symmetry** if it has both Quadrantal and Diagonal Symmetry, as shown in Figure $3.??(e)$, and therefore *must* satisfy

$$\begin{aligned} x(t_1, t_2) &\equiv x(-t_1, t_2) \equiv x(t_1, -t_2) \equiv x(-t_1, -t_2) \\ &\equiv x(t_2, t_1) \equiv x(-t_2, t_1) \equiv x(t_2, -t_1) \equiv x(-t_2, -t_1) \end{aligned} \tag{6.116}$$

The corresponding frequency domain symmetry is given by

$$X(j\omega_1, j\omega_2) \equiv X(-j\omega_1, j\omega_2) \equiv X(j\omega_1, -j\omega_2) \equiv X(-j\omega_1, -j\omega_2)$$
$$\equiv X(j\omega_2, j\omega_1) \equiv X(-j\omega_2, j\omega_1) \equiv X(j\omega_2, -j\omega_1) \equiv X(-j\omega_2, -j\omega_1) \quad (6.117)$$

6.4.3 REFLECTION SYMMETRIES OF THE MAGNITUDE SPECTRA OF REAL SIGNALS

In many practical situations, we are concerned with the reflection symmetries of *real* signals. When we come to consider the input-output behaviour of linear signal processing systems, for example, the signal $x(t)$ is often the real response of the system to some real excitation. In such cases, it is often possible to conclude that the magnitude spectrum $|X(j\omega)|$ possesses further symmetries in $dmn\ \omega$ that do not exist in $dmn\ \mathbf{t}$. These symmetries can then be exploited to save effort in the design of systems. Essentially, one uses $|X(j\omega)| = |X^*(j\omega)|$, the additional constraint in equation ( ).

Example 25 Twofold Symmetric Magnitude Spectrumof a 2D Real Signal

We will show that twofold symmetry of the magnitude spectrum is sufficent to ensure quadrantal symmetry of the magnitude spectrum of a real 2D signal.

Suppose that a real 2D signal has the twofold symmetry in the dimension t_1 so that

$$x(t_1, t_2) \equiv x(-t_1, t_2), \quad x(t_1, t_2) \in \mathbf{R}^2 \quad (6.118)$$

Then, by the general reflection Reflection Symmetry property,

$$X(j\omega_1, j\omega_2) = X(-j\omega_1, j\omega_2) \quad (6.119)$$

from which it follows that

$$|X(j\omega_1, j\omega_2)| = |X(-j\omega_1, j\omega_2)| \quad (6.120)$$

However, for real signals and from equation ( )

$$|X(j\omega_1, j\omega_2)| = |X(-j\omega_1, -j\omega_2)|$$

Then, comparing equation ( ) and equation ( ) it follows that

$$|X(j\omega_1, j\omega_2)| = |X(-j\omega_1, -j\omega_2)| = |X(-j\omega_1, j\omega_2)| = |X(j\omega_1, -j\omega_2)| \quad (6.121)$$

which proves quadrantal (four-fold axis) symmetry of the magnitude spectrum $|X(j\omega_1, j\omega_2)|$; that is, its twofold symmetric about n_1

implies twofold symmetry about n_2 and therefore quadrantal symmetry. Note that quadrantal symmetry does not imply 2D diagonal symmetry.

Example 26 Octant Symmetries of the Magnitude Spectrum of a Real 3D Signal

In $\mathbf{R}^3$, an there are eight octant regions, corresponding to the four quadrants in $\mathbf{R}^2$. In *dmn* **t**, they are the regions $\{\mathbf{t} | \pm t_1 \geq 0, \pm t_2 \geq 0, \pm t_3 \geq 0\}$. It follows from the congugate symmetry property in equation ( ) that, for real signals $x(t_1, t_2, t_3)$, the magnitude spectrum $|X(j\omega_1, j\omega_2, j\omega_3)|$ has reflective symmetry over each of the four pairs of opposite quadrants, as shown in Figure 3.??. For example, over the two opposite frequency domain octants $\{\omega | \omega_1 \leq 0, \omega_2 \geq 0, \omega_3 \geq 0\}$ and $\{\omega | \omega_1 \geq 0, \omega_2 \leq 0, \omega_3 \leq 0\}$, the magnitude spectrum $|X(j\omega_1, j\omega_2, j\omega_3)|$ has reflective symmetry.

6.4.4 SYMMETRIES IN THE DISCRETE DOMAIN CASE

It should be noted that discrete domain signals $x(\mathbf{n})$ cannot possess certain types of symmetries. For example, the rotational symmetries, $x(\mathbf{n}) = x(\mathbf{Rn}) \quad \mathbf{n} \in \mathbf{Z}^{\mathbf{m}}$, will only exist if $\mathbf{R}$ is such that $\mathbf{Rn} \in \mathbf{Z}^{\mathbf{m}}$. Clearly, if $\mathbf{R}$ corresponds to a 2D rotation of 90 degrees, the rotated samples $\mathbf{Rn}$ do indeed fall in $\mathbf{Z}^{\mathbf{m}}$, as shown in Figure 3.??(a). However, for the 2D rotation of 45 degrees, the rotated samples do not fall in $\mathbf{Z}^{\mathbf{m}}$, as shown in Figure \$3.??(b)\$, and therefore we cannot apply the symmetry result in equation ( ).

6.5 FREQUENCY DOMAIN PROPERTIES OF 3D LT SIGNALS

Consider the 3D LT **duration unbounded** signal $x_{lt}(\mathbf{t}), \mathbf{t} \in \mathbf{R}^3$, shown in Fig.??.?? and having the unit constant signal vector d and the 3D Fourier Transform $X_{lt}(\omega)$. Further, consider a second related LT signal $\hat{x}_{lt}(\mathbf{t})$ as shown in Figure.??.?? , which is obtained from $x_{lt}(\mathbf{t})$ by means of an appropriate rotation so that the direction of its constant signal intensity vector $\hat{\mathbf{d}}$ points in the direction of the t_3 axis; that is so that

$$\hat{\mathbf{d}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \tag{6.122}$$

as shown in Figure ??.?? . Let the 3D rotation that transforms $\mathbf{d}$ into $\hat{\mathbf{d}}$ be given by the geometry of Figure ??.?? so that

$$\mathbf{d} = \mathbf{R}\hat{\mathbf{d}} \tag{6.123}$$

where $\mathbf{R}$ is the 3D rotation matrix in equation ( ).

Let the 3D Fourier transform of $\hat{x}_{lt}(\mathbf{t})$ be written $\hat{X}_{lt}(\omega)$ so that

$$\hat{X}_{lt}(j\omega) = \int_{t=-\infty}^{\infty} \hat{x}_{lt}(\mathbf{t}) e^{-j\omega^{\mathrm{T}}\mathbf{t}} d\mathbf{t} \tag{6.124}$$

However, the signal $\hat{x}_{lt}(\mathbf{t})$ has constant intensity in the direction of the t_3 axis and therefore *can not be a function of* t_3. We may therefore write

$$\hat{x}_{lt}(\mathbf{t}) = x_{static}(t_1, t_2) \tag{6.125}$$

where $x_{static}(t_1, t_2)$ has been given in the subscript *static* to reflect the fact that, if t_3 is considered as the temporal axis, then the signal is static with respect to time. It is a 2D function of only the spatial axes t_1, t_2. From equations ( ) and ( ),

$$\hat{X}_{lt}(j\omega) = \int_{t=-\infty}^{\infty} x_{static}(t_1, t_2) e^{-j\omega^{\mathrm{T}}\mathbf{t}} d\mathbf{t} \tag{6.126}$$

Writing

$$e^{-j\omega^{\mathrm{T}}\mathbf{t}} = e^{-jw_1t_1} e^{-jw_2t_2} e^{-jw_3t_3} \tag{6.127}$$

in equation ( ) and separating the three integrations over each dimension gives

$$\hat{X}_{lt}(j\omega) = \int_{t_3=-\infty}^{\infty} e^{-jw_3t_3} dt_3 \int_{t_2=-\infty}^{\infty} \int_{t_1=-\infty}^{\infty} x_{static}(t_1, t_2) e^{-jw_1t_1} e^{-jw_2t_2} dt_1 dt_2 \tag{6.128}$$

Writing the Fourier transform of $x_{static}(t_1, t_2)$ as $\hat{X}_{static}(\omega_1, \omega_2)$, the above equation becomes

$$\hat{X}_{lt}(j\omega) = \hat{X}_{static}(j\omega_1, j\omega_2) \int_{t_3=-\infty}^{\infty} e^{-jw_3t_3} dt_3 \tag{6.129}$$

Now, the term involving the integration is recognized from the 1D Forurier transform pair

$$x(t_3) \equiv 1 \overset{1d}{\leftrightarrow} 2\pi\delta(\omega_3) = \int_{t_3=-\infty}^{\infty} e^{-jw_3t_3} dt_3 \tag{6.130}$$

Substituting the right side of equation ( ) into equation ( ) gives

$$\hat{X}_{lt}(j\omega) = X_{static}(j\omega_1, j\omega_2)2\pi\,\delta(\omega_3) \quad \textbf{(6.131)}$$

Therefore the spectrum $\hat{X}_{lt}(\omega)$ of a LT signal having a constant signal trajectory $\hat{d}$ in the direction of the t_3 axis is equal to $X_{static}(j\omega_1, j\omega_2)$ everywhere in the **planar** region $\omega_3 = 0$ and zero everywhere outside this plane. Clearly, $\hat{X}_{lt}(\omega)$ is **uniplanar** , as shown in Figure ??.?? , with the normal to the plane in the direction of the ω_3 axis.

It is now a simple matter to find the Fourier transform of the general LT signal $x_{lt}(t)$ by using the rotation property in equation ( ). From equations ( ), ( ) and ( ) ,

$$X_{lt}(j\omega) = \hat{X}_{lt}(j\mathbf{R}^{-\mathrm{T}}\omega) = \hat{X}_{lt}(j\mathbf{R}\omega) = X_{static}(j\omega_1{}', j\omega_2{}')2\pi\delta(\omega_3{}')$$

?j before omega 3?? **(6.132)**

where

$$\omega_1{}' = \cos\theta_1\omega_1 - \sin\theta_1\omega_2$$

$$\omega_2{}' = \cos\theta_2\sin\theta_1\omega_1 + \cos\theta_2\cos\theta_1\omega_2 - \sin\theta_2\omega_3$$

$$\omega_3{}' = \sin\theta_2\sin\theta_1\omega_1 + \sin\theta_2\cos\theta_1\omega_2 + \cos\theta_2\omega_3 \quad \textbf{(6.133)}$$

The dirac function $\delta(j\omega_3{}')$ implies that the Fourier transform $X_{lt}(j\omega)$ of the general LT signal $x_{lt}(t)$ is equal to $X_{static}(j\omega_1{}', j\omega_2{}')$ everwhere in the **planar region** $\omega_3{}' = 0$ and equal to zero everywhere outside of this planar region. Clearly, $\hat{X}_{lt}(\omega)$ **is a uniplanar signal**. The plane $\omega_3{}' = 0$ is written in terms of the above expression for $\omega_3{}' = 0$ as

$$(\sin\theta_2\sin\theta_1)\omega_1 + (\sin\theta_2\cos\theta_1)\omega_2 + (\cos\theta_2)\omega_3 = 0 \quad \textbf{(6.134)}$$

This plane is the region of support of the spectrum $X_{lt}(j\omega)$ of a general LT signal. The orientation of this plane is easily determined in terms of the orientation of $x_{lt}(t)$ in its domain. Equation ( ) implies that the rotation $\mathbf{R}$ in ω corresponds to the same rotation in $\mathbf{t}$; therefore the uniplanar spectrum $\hat{X}_{lt}(\omega)$ undergoes the same rotation into $\hat{X}_{lt}(\omega)$ as was applied to $x_{lt}(t)$ to obtain $\hat{x}_{lt}(t)$. We therefore expect that equation ( ) corresponds to the plane $\mathbf{d}^{\mathrm{T}}\omega = 0$, where $\mathbf{d}$ is the constant signal LT vector in the domain $\mathbf{t}$. This is easily proven from equations ( ), ( ) and ( ).

In summary, LT continuous domain signals $x_{lt}(t)$ possess a Fourier transform $X_{lt}(j\omega)$ that is uniplanar in a plane of the domain $\omega \in \mathbf{R}^3$ and the plane has a normal given by the constant signal vector $\mathbf{d}$ of the signal $x_{lt}(t)$.

The Finite Duration Case and Spectral Leakage Outside the Plane: It should be noted that our ideal analysis of the spectrum of an LT signal has assumed that $x_{lt}(t)$ has the LT property throughout the entire region $\mathbf{R}^\mathbf{m}$. In practical applications, the LT signal has a finite region of support along some (and usually all) of its dimensions. This has the practical consequence that the corresponding spectrum $\hat{X}_{lt}(\omega)$ is not exactly contained in a plane; there will be some leakage of the spectrum outside of the plane and this leakage will decrease as the region of support becomes larger in all of the dimensions.

6.6 FREQUENCY DOMAIN PROPERTIES OF PW SIGNALS

Assume the Fourier transform pair

$$X_{pw}(j\omega) \overset{md}{\Leftrightarrow} x_{pw}(\mathbf{t}) \tag{6.135}$$

for the ideal plane wave signal $x_{pw}(\mathbf{t})$, shown in Figure ?.?? for the 3D case. We want to determine the properties of the Fourier transform $X_{pw}(j\omega)$ of a plane wave. The following analysis is for the 3D case and is easily generalized to the mD case.

Consider the general 3D plane wave $x_{pw}(\mathbf{t})$ given by

$$x_{pw}(\mathbf{t}) \equiv x_{plane}(l) | \mathbf{d}^{\mathrm{T}}\mathbf{t} = l \tag{6.136}$$

implying the direction of propagation vector $\mathbf{d}$ shown in Figure ?.?? . Clearly, there exists another plane wave $\hat{x}_{pw}(\mathbf{t})$ having the direction of propagation

$$\hat{\mathbf{d}} \equiv [\hat{d}_1 \ \hat{d}_2 \ \hat{d}_3]^{\mathrm{T}} = [0 \ 0 \ 1]^{\mathrm{T}} \tag{6.137}$$

and related to $x_{pw}(\mathbf{t})$ by the appropriate rotation

$$\mathbf{d} = \mathbf{R}\hat{\mathbf{d}} \tag{6.138}$$

Then, we have

$$x_{pw}(\mathbf{t}) \equiv x_{plane}(l) | \hat{\mathbf{d}}^{\mathrm{T}}\mathbf{t} = l \tag{6.139}$$

or, equivalently,

$$\hat{x}_{pw}(\mathbf{t}) \equiv (x_{plane}(l) | t_3 = l) = x_{plane}(t_3) \tag{6.140}$$

where $\hat{x}_{pw}(\mathbf{t})$ is shown in Figure ?.?? .

First, we define the 3D Fourier transform of the plane wave $\hat{x}_{pw}(\mathbf{t}) = x_{plane}(t_3)$ as $\hat{X}_{plane}(\omega)$, so that by definition

$$\hat{X}_{plane}(\omega) = \int_{t=-\infty}^{\infty} x_{plane}(t_3) e^{-j\omega^T \mathbf{t}} d\mathbf{t}$$

$$= \int_{t_1=-\infty}^{\infty} e^{-j\omega_1 t_1} d\mathbf{t} \int_{t_2=-\infty}^{\infty} e^{-j\omega_2 t_2} dt_2 \int_{t_3=-\infty}^{\infty} x_{plane}(t_3) e^{-j\omega_3 t_3} dt_3$$

$$= 2\pi\delta(\omega_1) 2\pi\delta(\omega_2) \int_{t_3=-\infty}^{\infty} x_{plane}(t_3) e^{-j\omega_3 t_3} dt_3 \quad \textbf{(6.141)}$$

Now the term involving the 1D integration in the above equation is simply a 1D Fourier transform of the signal $x_{plane}(t_3)$ over the variable t_3 which, for brevity, we write as $X_{plane}(\omega)$ where

$$X_{plane}(\omega) \equiv \int_{t_3=-\infty}^{\infty} x_{plane}(t_3) e^{-j\omega_3 t_3} dt_3 \quad \textbf{(6.142)}$$

$X_{plane}(\omega)$ is simply the 1D Fourier transform of the signal in the direction of propagation of the plane wave. Combining the above two equations gives

$$\hat{X}_{pw}(\omega) = 2\pi\delta(\omega_1) 2\pi\delta(\omega_2) X_{plane}(\omega) \quad \textbf{(6.143)}$$

The product of the two delta functions ensure that the region of support of $\hat{X}_{pw}(\omega)$ is confined to the region of $\mathbf{R}^3$ defined by

$$\omega_1 = \omega_2 = 0 \quad \textbf{(6.144)}$$

which is simply **the straight line given by the** ω_3 **axis**. That is, as one might intuitively expect, **the plane wave** $\hat{x}_{pw}(\mathbf{t})$, having its direction of propagation along the t_3 axis, **has a Fourier transform that is zero everywhere outside the** ω_3 **axis in** $\omega \equiv \mathbf{R}^3$. We may now apply the rotation property of the Fourier transform to arrive at the required Fourier transform $X_{pw}(j\omega)$ of the general 3D plane wave. Thus, using the 3D rotation vector

$$\mathbf{R} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \cos\theta_2\sin\theta_1 & \cos\theta_2\cos\theta_1 & -\sin\theta_2 \\ \sin\theta_2\sin\theta_1 & \sin\theta_2\cos\theta_1 & \cos\theta_2 \end{bmatrix} \quad (6.145)$$

and the corresponding frequency domain rotational relationship

$$\omega = \mathbf{R}\hat{\omega} \quad (6.146)$$

we may write

$$X_{pw}(\omega) = 2\pi\delta(\hat{\omega}_1)2\pi\delta(\hat{\omega}_2)X_{plane}(\hat{\omega}_3) \quad (6.147)$$

where

$$\hat{\omega}_1 = (\cos\theta_1)\omega_1 - (\sin\theta_1)\omega_2$$

$$\hat{\omega}_2 = (\cos\theta_2\sin\theta_1)\omega_1 + (\cos\theta_2\cos\theta_1)\omega_2 - (\sin\theta_2)\omega_3$$

$$\hat{\omega}_3 = (\sin\theta_2\sin\theta_1)\omega_1 + (\sin\theta_2\cos\theta_1)\omega_2 + (\cos\theta_2)\omega_3 \quad (6.148)$$

Equation ( ) is the required result, expressing the Fourier transform of the general 3D plane wave $X_{pw}(\omega)$ in terms of the 1D Fourier transform $X_{plane}(\omega)$ of the signal $x_{plane}(l)$. The product $\delta(\hat{\omega}_1)\delta(\hat{\omega}_2)$ ensures that $X_{pw}(\omega)$ is zero everywhere outside of the rotated line defined by

$$\hat{\omega}_1 = \hat{\omega}_2 = 0$$

which is a line passing through the origin that is normal to the plane given by

$$\hat{\omega}_3 = 0$$

In terms of the frequency components of ω, this plane is given by

$\hat{\omega}_3 = (\sin\theta_2\sin\theta_1)\omega_1 + (\sin\theta_2\cos\theta_1)\omega_2 + (\cos\theta_2)\omega_3 = 0$ which is equivalent also to the plane

$\mathbf{d}^T\omega = 0$ with $\mathbf{d} = [(\sin\theta_2\sin\theta_1)\ (\sin\theta_2\cos\theta_1)\ (\cos\theta_2)]^t$, the direction of propagation of $x_{pw}(\mathbf{t})$ in its domain.

In summary, we have shown that a general 3D plane wave, given by equation ( ), and therefore characterized by $x_{planar}(l)$ and its direction of propagation $\mathbf{d}$,

has a 3D Fourier transform $X_{pw}(\omega)$ having a region of support that is a straight line through the origin. The direction of this line is determined by the fact that it is normal to the plane $\mathbf{d}^{1}\mathbf{t}$; therefore the direction cosines of the line are the components of $\mathbf{d}$. The value of $X_{pw}(\omega)$ along the straight line is given by the 1D Fourier transform $X_{planar}(\omega)$, which is the 1D Fourier transform of the 1D function $x_{planar}(l)$.

??in preceeding paragraph subscript ’planar’ is used but previously the subscript ’plane’ was used???