The Elimination of Spikes in the Magnitude Frequency Response of 2-D Discrete Filters by Increasing the Stability Margin

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Abstract—In the numerical design of two-dimensional (2-D) discrete filters, it is known that spurious spikes can and often do appear in the final optimized 2-D magnitude frequency response. Although such spikes usually have extraordinary narrow bandwidths, they can have large magnitudes and often lead to unsatisfactory spatial response. Two methods are presented to remove these spikes by increasing the stability margin in the regions of the frequency plane where they occur.

I. INTRODUCTION

RECENTLY, several numerical optimization methods have been presented for the design of recursive 2-D discrete filters. Some of them optimize the coefficients of the digital transfer function [6], [7] and others transform the problem to the design of a corresponding 2-D continuous transfer function using the double bilinear transformation [1]-[5]. Methods now exist [1]-[3] that guarantee BIBO-stability of the final optimized filter. Stability alone, however, is not enough to ensure satisfactory spatial-domain performance of the filter. Spikes in the magnitude frequency response often occur which causes undesirable transients in the filter response [3], [6]. In Fig. 1, for example, the optimized magnitude frequency response of a 3 × 3 low-pass digital filter is given (this filter is discussed in Example 1). The spikes in the frequency response cause an undesired large transient in the impulse response of the filter shown in Fig. 2. This large transient and the corresponding spikes in the frequency domain make the filter not suitable for processing signals with a significant amount of energy at the location of the spikes. We have found by experience that the numerical optimization process and, in particular, the objective function are insensitive to the presence of spikes because the spikes are sufficiently narrow in bandwidth that they occur between the surrounding optimization grid points. Attempts to eliminate these spikes by clustering more grid points around them and/or weighting the grid points more heavily are unsuccessful because the spikes change shape or migrate to other locations during optimization.

The stability margin for 2-D discrete systems is defined as the shortest distance between a singularity of a filter (a 2-D manifold) and the stability limit (the boundary of the unit bidisc) [8], [10]. A spike is indicative of a small distance of a singularity from $T^{2}$, the distinguished boundary of the unit bidisc, at the spike location and, consequently, of a small stability margin at this location [3], [6]. In this paper, a method is presented to eliminate spikes by increasing the stability margin of the filter. This is done in one of two ways depending on the location of the spikes in the frequency plane. It is our experience that spikes usually occur close to the line $\omega_1 = - \omega_2$ and well removed from the origin in the frequency plane. The frequency-dependent method proposed in Section III, can be applied in such cases. The spikes are removed by increasing the stability margin in the region well removed from the origin of the frequency plane where the spikes are known to be located. Near the center of the frequency plane, the stability margin remains almost unchanged and, therefore, the selectivity of the filter in this region is not significantly affected. In the case where the spikes occur anywhere in the 2-D frequency plane, not just well removed from the origin, a frequency-independent method can be applied to increase the stability margin equally for all frequencies.

In Section II, the recursive filter design method in [1]-[3] is reviewed and in Section III this method is modified to remove spikes by increasing the stability margin. Two examples are presented in Section IV to illustrate the elimination of two different spike distributions.

II. REVIEW OF THE BIBO-STABLE DESIGN METHOD IN [1]-[3]

In [1]-[3], a numerical optimization method is presented to design BIBO-stable 2-D digital filters approximating a specified frequency response. The method is based on designing a 2-D continuous transfer function $T(s_1, s_2)$ which corresponds to the discrete one $G(z_1, z_2)$, after double bilinear transformation:

$$T(s_1, s_2) = G(z_1, z_2)|_{z_1, z_2} = (1 - n_1 z_1 / n_2) \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{ij} s_1^i s_2^j$$

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(1)
In order to ensure stability of the continuous transfer function, the coefficients \( \{ p_{ij} \} \) and \( \{ q_{ij} \} \) are not optimized directly; rather, a different set of optimization variables is chosen. A lossless frequency-independent multiport network terminated in \( n_1 \) unit capacitors \( 1/s_1 \) and \( n_2 \) unit capacitors \( 1/s_2 \) is considered and the parameters of its admittance matrix \( \{ y_{ij} \} \) are used as the optimization variables. The voltage transfer function of the network or its driving-point admittance function is considered as the continuous transfer function \( T(s_1, s_2) \). \( T(s_1, s_2) \) has a strictly Hurwitz denominator polynomial for all real values of \( \{ y_{ij} \} \), as shown in [1]–[3]. The relationships between the coefficients \( \{ p_{ij} \} \) and \( \{ q_{ij} \} \) of (1) and \( \{ y_{ij} \} \) depend on the structure of the network considered. Two different structures are used and are based on the direct passive equiterminated network method [3] and the determinant assignment method [1], [2]. Both methods are summarized in Appendix A. The advantage of the equiterminated method is that the magnitude frequency response \( |G(e^{j\omega_1}, e^{j\omega_2})| \) has an upper bound of 1 and, therefore, spikes can not have a magnitude larger than 1 [3]. When the determinant assignment method is used, spikes can occur with very large magnitudes [3]. The determinant assignment method, however, usually produces more selective filters.

Numerical optimization with \( \{ y_{ij} \} \) as the unconstrained optimization variable is carried out using the Fletcher–Powell method. A weighted least squares error criterion is used as the objective function. The optimization grid and the desired frequency-response specifications are selected in the discrete frequency plane. The discrete grid is then transformed into the continuous grid using the double bilinear transformation. The minimum of the objective function with respect to the parameters \( \{ y_{ij} \} \) is obtained by evaluating the objective function at the continuous \( (\Omega_1, \Omega_2) \) grid points. From the resulting continuous transfer function having a strictly Hurwitz denominator polynomial, the discrete version is obtained using the double bilinear transformation:

\[
s_i = \frac{1 - z_i}{1 + z_i}, \quad i = 1, 2.
\]

The discrete transfer function is BIBO-stable except in the case where nonessential singularities of the second kind arise at the distinguished boundary of the unit bidisc due to discontinuities of the continuous transfer function at \( s_1 = \infty \) and/or \( s_2 = \infty \) (see [9]).

### III. METHODS FOR REMOVING SPIKES BY INCREASING STABILITY MARGIN

The cause of the spikes is the presence of singularities of \( T(s_1, s_2) \) close to the region

\[
H^2 = \left\{ (s_1, s_2) \mid \text{Re}(s_1) = 0 \text{ and } \text{Re}(s_2) = 0 \right\}.
\]

To remove these spikes, the distance of the singularities of \( T(s_1, s_2) \) to \( H^2 \) must be increased.

The methods discussed in this section require the existence of a design technique, such as that described in Section II, for obtaining a 2-D continuous transfer function \( T(s_1, s_2) \) having the required frequency response and a strictly Hurwitz denominator. This ensures that all singularities of \( T(s_1, s_2) \) are excluded from the region \( \{(s_1, s_2) | \text{Re}(s_1) \geq 0 \text{ and } \text{Re}(s_2) \geq 0 \} \). The essence of the method is to modify the continuous domain optimization algorithm so that the magnitude frequency response specifications are met not on \( H^2 \), but on an alternate optimization surface, which is inside the region \( \{(s_1, s_2) | \text{Re}(s_1) \geq 0 \text{ and } \text{Re}(s_2) \geq 0 \} \) and well removed from \( H^2 \). The discrete transfer function is then obtained using a transformation which transforms the alternate optimization surface to the distinguished boundary of the unit bidisc. In the case where a conventional algorithm, such as the one in Section II, leads to spikes in the magnitude frequency response, it is proposed that the modified algorithm described here is used for the elimination of such spikes by further optimization. The transfer function obtained with the conventional algorithm is used as the initial design and the frequency specifications are met on an alternate optimization surface. The resulting design satisfies the frequency response specifications and has a larger stability margin than the transfer function obtained with the conventional algorithm. Depending on which optimization surface and transformation are used, either a different increase of the stability margin.
for each frequency point (frequency-dependent method), or
a uniform increase of the stability margin (frequency-independent method) will result.

A. The Frequency-Dependent Method

Consider the transformation

\[ s_i = \frac{1 + \alpha}{2} \left( \frac{1 - z_i}{1 + \alpha z_i} \right), \quad i = 1, 2 \]  

(4)

where

\[ 0 < \alpha < 1. \]  

(5)

Equation (4) transforms the unit circle of the z-plane given by

\[ T = \{ z ||z| = 1 \} \]  

(6)

into a right half s-plane circle, having center at \((1 + a/2, 1 - a), 0\) and passing through the origin:

\[ H_a = \{ s ||s = u_i + jw_i \text{ where } u_i = \frac{1 + a}{2 - 2a}, w_i = \left( \frac{1 + a}{2 - 2a} \right)^2 \} \]  

(7)

as shown in Fig. 3. The inverse transformation of (4) is given by

\[ z_i = \frac{1 + a}{1 + \alpha} s_i, \quad i = 1, 2. \]  

(8)

Equation (8) transforms the imaginary axis of the s-plane into a circle in the z-plane having center at \((1 - a/2a, 0)\) and passing through \((-1/a, 0)\), given by

\[ T_a = \{ z ||z = x_i + jy_i \text{ where } x_i = \frac{1 - a}{2a}, y_i = \left( \frac{1 - a}{2a} \right)^2 \} \]  

(9)

Obviously (8) transforms the s-plane circle \(H_a\), (7), into the unit circle \(T\), (6), in the z-plane. (7) and (9) are derived in Appendix B.

In order to increase the stability margin using the frequency-dependent method, the design algorithm is modified by replacing the double bilinear transformation (2) with the transformation (4). This implies the following changes in the algorithm. The discrete frequency plane given by

\[ T^2 = \{(z_1, z_2)||z_1| = |e^{j\omega_1}| = 1, \quad |z_2| = |e^{j\omega_2}| = 1\} \]  

(10)

is transformed using (4) into

\[ H_a^2 = \{(s_1, s_2)|s_1 = u_1 + jw_1, s_2 = u_2 + jw_2 \text{ where } u_i = \frac{1 + a}{2 - 2a}, w_i = \left( \frac{1 + a}{2 - 2a} \right)^2, i = 1, 2\} \]  

(11)

which is different than the continuous frequency plane \(H^2\), (3), obtained using the double bilinear transformation (2). The superscript 2 indicates the 2-D case.

The optimization is now carried out by evaluating the objective function on \(H_a^2\), and the resulting 2-D continuous transfer function \(T(s_1, s_2)\) is transformed using (4) to the corresponding digital transfer function \(G(z_1, z_2)\).

As mentioned before, the denominator polynomial of \(T(s_1, s_2)\) is strictly Hurwitz due to properties of the network structure (see Appendix A). The strictly Hurwitz denominator polynomial and the fact that (8) transforms \(H^2\) into

\[ T_a^2 = \{(z_1, z_2)|z_1 = x_1 + jy_1, z_2 = x_2 + jy_2 \text{ where } x_i = \frac{1 - a}{2a}, y_i = \left( \frac{1 - a}{2a} \right)^2, i = 1, 2\} \]  

(12)

imply that \(G(z_1, z_2)\) will not only have all singularities outside the unit bidisc, but also that for every discrete 2-D frequency \((\omega_1, \omega_2)\), the stability margin is increased by a value which varies from 0 at \((\omega_1, \omega_2) = (0, 0)\) to \((1 - a)/a\) at \((\omega_1, \omega_2) = (\pm \pi, \pm \pi)\). This method eliminates spikes well removed from the origin of the discrete frequency plane (by increasing the stability margin there) without affecting the selectivity of the filter at low frequencies (where stability margin is almost unchanged). This result is demonstrated in Section IV by means of an example.

The amount by which the stability margin is increased depends on the choice of the parameter \(a\) in (4). In the special case where \(a = 1\), (4) becomes the double bilinear transformation (2) so that \(H^2 = H_{2\pi}^2\). In this case, obviously, the stability margin remains unchanged everywhere in the 2-D discrete frequency plane.

B. Frequency-Independent Method

Consider the transformation

\[ s_i = \frac{1 - \xi z_i}{1 + \xi z_i}, \quad i = 1, 2 \]  

(13)

where

\[ \xi = \frac{1}{1 + \alpha}, \quad \alpha > 0. \]  

(14)
Equation (13) transforms the unit circle $T$ of the $z$-plane, (6), into the circle

$$H_s = \left\{ s \mid s = u + jw \text{ where } \left( \frac{u - 1 + \xi^2}{1 - \xi^2} \right)^2 + w^2 = \left( \frac{2\xi}{1 - \xi^2} \right)^2 \right\}$$  \hspace{1cm} (15)

in the $s$-plane, as shown in Fig. 4. The inverse transformation of (13) is given by

$$z_i = \frac{1 - s_i}{\xi + s_i}, \quad i = 1, 2.$$  \hspace{1cm} (16)

Equation (16) transforms the imaginary axis of the $s$-plane into the circle:

$$T_o = \left\{ z \mid |z| = 1 + \sigma \right\}$$  \hspace{1cm} (17)

in the $z$-plane and the circle $H_o$ of the $s$-plane, (15), into the unit circle $T$ in the $z$-plane.

The use of transformation (13) instead of the double bilinear transformation implies the following modifications in the design algorithm. The discrete frequency plane $T^*$, (10) is transformed using (13) into

$$H_x^* = \left\{ (s_1, s_2) \mid s_1 = u_1 + jw_1, \quad s_2 = u_2 + jw_2 \text{ where } \left( \frac{u_i - 1 + \xi^2}{1 - \xi^2} \right)^2 + w_i^2 = \left( \frac{2\xi}{1 - \xi^2} \right)^2, \quad i = 1, 2 \right\}. \hspace{1cm} (18)$$

The optimization is carried out by evaluating the objective function on $H_x^*$, and the resulting continuous transfer function is transformed using (13) to the corresponding discrete filter $G(z_1, z_2)$.

The denominator polynomial of $T(s_1, s_2)$ is known to be strictly Hurwitz and (16) transforms $H_o^2$, (2), into

$$T_o^2 = \left\{ (z_1, z_2) \mid |z_1| = 1 + \sigma, \quad |z_2| = 1 + \sigma \right\}. \hspace{1cm} (19)$$

It follows then that $G(z_1, z_2)$ has not only all singularities outside the unit circle, but also has a stability margin for every $(\omega_1, \omega_2)$ which is greater than $\sigma$. This method, therefore, increases the stability margin by a constant amount $\sigma$ for all frequencies and can be used to remove spikes which are anywhere in the entire frequency plane.

The modified algorithm needs to be used only in the case where spikes appear in the frequency response of a transfer function after optimization using the original algorithm. In such a case the design obtained from the original algorithm can be used as initial design for further optimization using the modified algorithm to remove spikes and meet the frequency response specifications. The number of the optimization steps depends on how "close" $H^2_x$ or $H^2_x$ are to $H^2$, or in other words, how much the stability margin has to be increased. In some cases where the magnitude of the spikes is small compared to the magnitude of the passband and only a small increase of the stability margin is required, the use of transformations (4) or (13) would be sufficient to remove the spikes without any further optimization and without affecting the frequency response.

The methods (i) and (ii) above are not restricted to the design algorithm in Section II, but apply to any 2-D recursive filter optimization algorithm which guarantees and BIBO-stability of the optimized filter. The double bilinear transformation (2) can be replaced by the transformations outlined (i) and (ii), or other transformations can be found to transform the discrete grid to another optimization grid which satisfies the stability margin requirements.

In [6], a 2-D filter design algorithm based on numerical optimization is presented. Stability margin constraints are considered there by adding to the objective function a penalty function which depends on the stability margin. The resulting filter, in the case where it is BIBO-stable, corresponds to the results obtained by the frequency-independent method presented here.

IV. EXAMPLES

In this section, two examples are presented which illustrate the advantages and disadvantages of each method in Section III.

Example 1

A circularly symmetric low-pass filter of order $(3, 3)$ is to be designed. The stopband radius is $\pi/4$ and the specified magnitude frequency response is of a Butterworth type. An equiterminated network structure (see Appendix A) is used. Fig. 1 shows the frequency response and Fig. 2 the impulse response of the optimized filter when no stability margin constraints are considered. The frequency response has a satisfactory passband, but spikes occur in the stopband which cause the unsatisfactory transient response. These spikes are located around $(\omega_1, \omega_2) = (\pm 135^\circ, \pm 135^\circ)$. They are eliminated by increasing the stability margin of the filter using the methods outlined in Section III:

i) frequency-dependent, with $a = 0.8$,

ii) frequency-independent, with $\sigma = 0.21$.

For purposes of comparison, the values of $a$ and $\sigma$ are selected in each case so that they cause the same stability margin increase for $(\omega_1, \omega_2) = \pm 135^\circ$. They are eliminated by increasing the stability margin of the filter using the methods outlined in Section III:

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The optimization is carried out by evaluating the objective function on $H^2_x$, and the resulting continuous transfer function is transformed using (13) to the corresponding discrete filter $G(z_1, z_2)$.

The denominator polynomial of $T(s_1, s_2)$ is known to be strictly Hurwitz and (16) transforms $H^2_o$, (2), into

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The modified algorithm needs to be used only in the case where spikes appear in the frequency response of a transfer function after optimization using the original algorithm. In such a case the design obtained from the original algorithm can be used as initial design for further optimization using the modified algorithm to remove spikes and meet the frequency response specifications. The number of the optimization steps depends on how "close" $H^2_x$ or $H^2_x$ are to $H^2$, or in other words, how much the stability margin has to be increased. In some cases where the magnitude of the spikes is small compared to the magnitude of the passband and only a small increase of the stability margin is required, the use of transformations (4) or (13) would be sufficient to remove the spikes without any further optimization and without affecting the frequency response.

The methods (i) and (ii) above are not restricted to the design algorithm in Section II, but apply to any 2-D recursive filter optimization algorithm which guarantees and BIBO-stability of the optimized filter. The double bilinear transformation (2) can be replaced by the transformations outlined (i) and (ii), or other transformations can be found to transform the discrete grid to another optimization grid which satisfies the stability margin requirements.

In [6], a 2-D filter design algorithm based on numerical optimization is presented. Stability margin constraints are considered there by adding to the objective function a penalty function which depends on the stability margin. The resulting filter, in the case where it is BIBO-stable, corresponds to the results obtained by the frequency-independent method presented here.

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It follows then that $G(z_1, z_2)$ has not only all singularities outside the unit circle, but also has a stability margin for every $(\omega_1, \omega_2)$ which is greater than $\sigma$. This method, therefore, increases the stability margin by a constant amount $\sigma$ for all frequencies and can be used to remove spikes which are anywhere in the entire frequency plane.

The modified algorithm needs to be used only in the case where spikes appear in the frequency response of a transfer function after optimization using the original algorithm. In such a case the design obtained from the original algorithm can be used as initial design for further optimization using the modified algorithm to remove spikes and meet the frequency response specifications. The number of the optimization steps depends on how "close" $H^2_x$ or $H^2_x$ are to $H^2$, or in other words, how much the stability margin has to be increased. In some cases where the magnitude of the spikes is small compared to the magnitude of the passband and only a small increase of the stability margin is required, the use of transformations (4) or (13) would be sufficient to remove the spikes without any further optimization and without affecting the frequency response.

The methods (i) and (ii) above are not restricted to the design algorithm in Section II, but apply to any 2-D recursive filter optimization algorithm which guarantees and BIBO-stability of the optimized filter. The double bilinear transformation (2) can be replaced by the transformations outlined (i) and (ii), or other transformations can be found to transform the discrete grid to another optimization grid which satisfies the stability margin requirements.

In [6], a 2-D filter design algorithm based on numerical optimization is presented. Stability margin constraints are considered there by adding to the objective function a penalty function which depends on the stability margin. The resulting filter, in the case where it is BIBO-stable, corresponds to the results obtained by the frequency-independent method presented here.
frequency and impulse response of the optimized filter using the frequency-independent method. As expected, the spikes in the frequency response and the large transient in the impulse response are eliminated in both cases. However, the resulting filter using the frequency-independent method has a reduced selectivity in the transition region, while the selectivity of the resulting filter using the frequency-dependent method is not affected.

**Example 2**

A fan filter is to be designed with a 90° wide passband. The magnitude frequency response specification is 0 in the stopband and 1 in the passband. The determinant assignment method (see Appendix A) is used. Fig. 9 shows the frequency response of the filter designed without stability margin constraints. Spikes occur along the 45° line. These spikes are removed using the following methods:

i) frequency dependent, with $a = 0.96$,

ii) frequency independent, with $\sigma = 0.02$.

The filter of Fig. 9 is used as the initial design in both cases. Figs. 10 and 11 show the magnitude frequency response of the resulting filter without further optimization using transformations (4) and (13) corresponding to the frequency-dependent and frequency-independent methods, respectively. With the frequency-dependent method, only the spikes well removed from the origin are removed; the spikes at low frequencies remain unaffected. The frequency-independent method gives better results in this example. As can be seen in Fig. 11, the spikes are eliminated for all frequencies.
Consider a lossless \((2+n_1+n_2, 2+n_1+n_2)\)-port network which is terminated in a unit resistance, \(n_1\) unit capacitors \(1/s_1\) and \(n_2\) unit capacitors \(1/s_2\), Fig. 12.

It is shown in [1], [2] that the input admittance \(Y_{in}(s_1, s_2)\) has a denominator polynomial \(\Delta(s_1, s_2)\) of order \((n_1, n_2)\) which is strictly Hurwitz. \(\Delta(s_1, s_2)\) is given by

\[
\Delta(s_1, s_2) = \det [Y(s_1, s_2)] \quad (A1)
\]

where \(Y(s_1, s_2) = \{Y_{ij}\}\) is a \((1+n_1+n_2) \times (1+n_1+n_2)\) matrix satisfying

\[
\begin{align*}
Y_{ij} &= -Y_{ji}, \quad i \neq j \\
Y_{11} &= 1 \\
Y_{ii} &= s_1, \quad i = 2, 3, \cdots, (n_1+1) \\
Y_{ii} &= s_2, \quad i = (n_1+2), \cdots, (n_1+n_2+1). \quad (A2)
\end{align*}
\]

\(\{Y_{ij}, i \neq j\}\) are the elements of the networks admittance matrix. Using the property that \(\Delta(s_1, s_2)\) is strictly Hurwitz for every real value of \(\{Y_{ij}, i \neq j\}\), it follows that a continuous transfer function \(T(s_1, s_2)\) having \(\Delta(s_1, s_2)\) as a denominator polynomial will result in a BIBO-stable 2-D discrete filter after double bilinear transformation. (It is assumed that \(T(s_1, s_2)\) has no discontinuities at \(s_1\) and/or \(s_2 = \infty\).) The continuous function is, therefore, selected as

\[
T(s_1, s_2) = \frac{P(s_1, s_2)}{\Delta(s_1, s_2)} \quad (A3)
\]

and the polynomials \(P(s_1, s_2)\) and \(\Delta(s_1, s_2)\) are obtained by numerical optimization where \(\{P_{ij}\}\) and \(\{Y_{ij}, i \neq j\}\) are the unconstrained optimization variables.

**B. Direct Passive Equiterminated Network Method [3]**

Consider a \((2+n_1+n_2, 2+n_1+n_2)\)-port network which is terminated in two unit resistors, \(n_1\) unit capacitors \(1/s_1\) and \(n_2\) unit capacitors \(1/s_2\), Fig. 13. The analog transfer function \(V_2(s_1, s_2)/V_1(s_1, s_2)\) is equated directly to the analog transfer function \(T(s_1, s_2)\). In [3], it is shown that

\[
T(s_1, s_2) = \frac{V_2(s_1, s_2)}{V_1(s_1, s_2)} = \frac{\Delta_{12}(s_1, s_2)}{\Delta(s_1, s_2)} \quad (A4)
\]

where

\[
\Delta(s_1, s_2) = \det [\hat{Y}(s_1, s_2)] \quad (A5)
\]
Fig. 13. A 2-D equiterminal equiterminated \((2+n_1+n_2)\)-port lossless network.

and

\[
\Delta_{12}(s_1, s_2) = \text{determinant of } \hat{Y} \text{ with row 1 and column 2 removed.}
\]

\[
\hat{Y} = \{Y_{ij}\} \text{ is a } (2+n_1+n_2) \times (2+n_1+n_2) \text{ matrix which satisfies}
\]

\[
\begin{align*}
Y_{ij} &= -Y_{ji}, \quad i \neq j \\
Y_{ii} &= 1, \quad i = 1,2 \\
Y_{ii} &= s_1, \quad i = 3, 4, \ldots (n_1+2) \\
Y_{ii} &= s_2, \quad i = (3+n_1), \ldots, (2+n_1+n_2).
\end{align*}
\]

\(\Delta(s_1, s_2)\) is a strictly Hurwitz polynomial for every real value of \(\{Y_{ij}, i \neq j\}\) [3] and, therefore, the resulting 2-D digital filter after double bilinear transformation is stable. The continuous transfer function \(T(s_1, s_2)\) can be obtained by numerical optimization where \(\{Y_{ij}, i \neq j\}\) are the unconstrained optimization variables.

**APPENDIX B**

i) Equation (7): The transformation (4) applied to the unit circle \(T\) gives

\[
e = \frac{1+a}{2} - j\frac{e^{j\omega}}{2 + 1 + e^{j\omega}}
\]

\[
= \frac{(a+1)(1-a)(1-\cos \omega)}{2(1+a^2+2a\cos \omega)} - j\frac{(a+1)^2 \sin \omega}{2(1+a^2+2a\cos \omega)}
\]

which can be written as

\[
s = u_a + jw_a.
\]

Equation (7) can be verified by substituting \(u_a\) and \(w_a\) in it.

ii) Equation (9): By applying transformation (8) to the imaginary axis of the \(s\)-plane one gets

\[
z = \frac{1+a}{2} - j\Omega \left( \frac{1+a}{2} - a\Omega^2 \right) - \frac{\Omega (1+a)^2}{2}
\]

\[
= \frac{1+a}{2} + aj\Omega \left( \frac{1+a}{2} + a^2\Omega^2 \right) - j\frac{(1+a)^2}{2} + a^2\Omega^2
\]

or

\[
z = x_a + jy_a.
\]

Equation (9) can be verified by substituting \(x_a\) and \(y_a\) in it.

**References**


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