directed towards design of inherently concurrent lossless decoders and Viterbi-type decoders. Such decoders should be capable of operating at high speed and should require marginal complexity increase as compared with the sequential decoders.

REFERENCES


Applications of Complex Filters to Realize Three-Dimensional Combined DFT/LDE Transfer Functions

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Abstract—The recently proposed method of combined discrete Fourier transform (DFT)/linear difference equation (LDE) filtering of three-dimensional (3-D) spatio-temporal signals employs LDE’s with real coefficients. It is shown here that the real-coefficient LDE’s impose transfer function symmetries that prevent the 3-D filter from distinguishing between 3-D input signals having reflective symmetry in the direction of the LDE filtering. These transfer function symmetry constraints are relaxed by using complex-coefficient LDE’s.

I. INTRODUCTION

The method of combined discrete Fourier transform (DFT)/linear difference equation (LDE) filtering has been proposed for the enhancement of multidimensional (MD) digital signals [1]. As shown in Fig. 1, the method operates on a 3-D input signal x(n₁, n₂, n₃) by applying the two-dimensional (2-D) DFT on x(n₁, n₂, n₃), performing 1-D LDE filtering of the resulting complex 2-tuple frequency points and then by applying the 2-D inverse DFT to the outputs of the LDE’s. This method has applications in the enhancement of dynamic 2-D images.

The combined DFT/LDE method employs LDE’s that have real coefficients. In this paper, we show for the 3-D case that LDE’s with real coefficients impose a symmetry constraint on the frequency response H(jω₁, jω₂, jω₃) that can be relaxed by employing LDE’s having complex coefficients. Assuming spatial variables n₁, n₂ and a temporal variable n₃, with LDE filtering in the n₃ direction [1], LDE’s having real coefficients lead to output images y(n₁, n₂, n₃) in Fig. 1 having identical responses for both x(n₁, n₂, n₃) and the time-reversed image x(n₁, n₂, −n₃). It is shown here that this limitation may be overcome by employing LDE’s with complex coefficients.

II. TRANSFER FUNCTION SYMMETRY IMPOSED ON H(jω₁, jω₂, jω₃) BY LDE’S HAVING REAL COEFFICIENTS

We assume combined DFT/LDE filters where the 2-D DFT is performed over ω₁ and ω₂ (ω₁, ω₂ ∈ R²) and 1-D continuous frequency domain (LDE) filtering in the direction n₃ (ω₃ ∈ C¹). The overall input-output frequency response of the 3-D combined 2D-DFT/LDE filter is written as H(jω₁, jω₂, jω₃) = M(ω₁, ω₂, ω₃) where ω₁, ω₂, ω₃ belong to the 3-D frequency space R³ that is discrete in ω₁, ω₂ and continuous in ω₃; that is, R³ = Z³ × B. For real input signals x(n₁, n₂, n₃) and identical real-coefficient LDE’s at each pair of 2-tuples (ω₁, ω₂) and (−ω₁, −ω₂), the DFT operation imposes the following symmetry constraints [2]:

\[ H(jω₁, jω₂, jω₃) = H(−jω₁, −jω₂, jω₃) \]  \hspace{1cm} (1)

implying magnitude centro-symmetry (MCS) and phase centro-anti-symmetry (PCAS) [3] over (ω₁, ω₂). The output signal y(n₁, n₂, n₃) is real [1]. Consider the eight octants in R³. From (1), the DFT operation constrains them into four pairs of MCS/PCAS octants, as follows:

\[ H(jω₁, jω₂, jω₃) = H(−jω₁, −jω₂, jω₃) \]
\[ H(jω₁, jω₂, −jω₃) = H(−jω₁, −jω₂, −jω₃) \]
\[ H(jω₁, −jω₂, jω₃) = H(−jω₁, jω₂, jω₃) \]
\[ H(jω₁, −jω₂, −jω₃) = H(−jω₁, jω₂, −jω₃) \]

in R³ due to DFT operations.

If the LDE filters on ω₃ have real coefficients, the following additional symmetry constraint applies to ω₃:

\[ H(jω₁, jω₂, jω₃) = H(jω₁, jω₂, −jω₃) \]  \hspace{1cm} (2)

additional octant constraints in R³ due to real LDE coefficients.

Combining (2) and (3), H(jω₁, jω₂, jω₃) is constrained as fol-
Thus, only two of the eight octants are independent, and the remaining six are constrained by symmetry. Although (4) allows the magnitude response \( M(\omega_1, \omega_2, \omega_3) \) to have quadrantal symmetry in \((\omega_1, \omega_2)\), it does enforce the following type of symmetry over \(\omega_3\) that is undesirable in some practical situations.

Consider an input signal \( x(n_1, n_2, n_3) \) and a second, different input signal \( x_l(n_1, n_2, n_3) \) such that \( x_l(n_1, n_2, n_3) = x(n_1, n_2, \omega_3) \), corresponding to a time-reversed version of \( x(n_1, n_2, n_3) \). We therefore have

\[
y_l(n_1, n_2, n_3) = Y_l(j\omega_1, j\omega_2, j\omega_3) = X(j\omega_1, j\omega_2, j\omega_3) H(j\omega_1, j\omega_2, j\omega_3).
\]

The DFT has the property that \( X_l(j\omega_1, j\omega_2, j\omega_3) = X(j\omega_1, j\omega_2, j\omega_3) \) so that \( Y_l \) may be written as

\[
y_l(n_1, n_2, n_3) = X_j(j\omega_1, j\omega_2, j\omega_3) H(j\omega_1, j\omega_2, j\omega_3).
\]

From (3) and (6), we may write

\[
y_l(j\omega_1, j\omega_2, j\omega_3) = X(j\omega_1, j\omega_2, j\omega_3) H(j\omega_1, j\omega_2, j\omega_3).
\]

and therefore, from (5a) and (5b),

\[
y_l(j\omega_1, j\omega_2, j\omega_3) = Y(j\omega_1, j\omega_2, -j\omega_3).
\]

Taking the inverse 2-D DFT of (8) over \((\omega_1, \omega_2)\) and then the inverse Fourier transform over \(\omega_3\) leads directly to the required result that

\[
y_l(n_1, n_2, n_3) = y(n_1, n_2, -n_3).
\]

implying that, for a 3-D combined DFT/LDE filter with real-coefficient LDE’s, time-reversal of the input leads to a corresponding time-reversal of the output.

It is interesting to note that real-coefficient 3-D LDE filters and the 3-D DFT do not suffer from this limitation because they are only constrained as follows:

\[
H(j\omega_1, j\omega_2, j\omega_3) = H(-j\omega_1, -j\omega_2, -j\omega_3), \quad \omega_2 = \omega_2,
\]

implying four independent octants having MCS/PCAS in \((\omega_1, \omega_2, \omega_3)\).

Example: 3-D Frequency-Planar (FP) Filters using Real Coefficient LDE’s [4]

Frequency-planar (FP) filters have been used to selectively enhance linear-trajectory spatio-temporal signals, using real-coefficient LDE’s with identical LDE’s at each pair of 2-tuples \((\omega_1, \omega_2)\) and \((-\omega_1, -\omega_2)\). They are implemented using narrow-band bandpass LDE’s having center frequencies \(\omega_3\) where the passband surrounds the plane \[4\]

\[
\alpha_1 \omega_1 + \alpha_2 \omega_2 + \alpha_3 \omega_3 = 0 \quad P_1.
\]

The magnitude transfer function \( M(\omega_1, \omega_2, \omega_3) \) is unity on \( P_1 \) and ideally rolls off rapidly to zero outside \( P_1 \). However, according to (3), there must exist a second passband plane

\[
\alpha_1 \omega_1 + \alpha_2 \omega_2 - \alpha_3 \omega_3 = 0 \quad P_2
\]

because \( M(\omega_1, \omega_2, \omega_3) = M(\omega_1, \omega_2, -\omega_3) \). If the passband \( P_1 \) selectively transmits the linear trajectory signal \( x(n_1, n_2, n_3) \), then the passband \( P_2 \) selectively transmits the time-reversed signal \( x(n_1, n_2, -n_3) \). Ideally, we would like to remove the second passband \( P_2 \) because we often want to enhance a linear-trajectory signal, but not a signal on the time-reversed trajectory.

III. THE DESIGN AND APPLICATION OF COMPLEX-COEFFICIENT LDE’s

It is now shown that combined DFT/LDE filters can be designed so that \( H(j\omega_1, j\omega_2, j\omega_3) \) has four independent octants by allowing the DFT filters to have complex coefficients. Let \( T(j\omega_1, j\omega_2, j\omega_3) \) represent the frequency response of the 1-D complex-coefficient LDE filter on \( \omega_3 \) at the 2-tuple \((\omega_1, \omega_2)\). We select complex-coefficient LDE’s at each pair of 2-tuples \((\omega_1, \omega_2)\) and \((-\omega_1, -\omega_2)\) such that the corresponding frequency responses over \(\omega_3\) are complex conjugates; that is, so that

\[
T(j\omega_1, j\omega_2, j\omega_3) = T(-j\omega_1, -j\omega_2, -j\omega_3).
\]

Therefore, (2) remains in effect but (3) does not apply, and four of the eight octant are independent.

Complex Bandpass LDE Filters

Using complex-coefficient LDE’s, we often require complex bandpass filters \( T(j\omega_1, j\omega_2, j\omega_3) \) at each 2-tuple \((\omega_1, \omega_2)\) having the idealized magnitude response \( M(\omega_1, \omega_2, \omega_3) \) shown in Fig. 2.
A straightforward approach to the design of these 1-D complex filters is to employ the following discrete-domain method (which is equivalent to that described in [5] for the case of continuous-domain complex filters). We choose a suitable discrete-domain real-coefficient LDE corresponding to the low-pass filter \( T(z) \), expressed in the factored form as

\[
T(z) = \frac{1}{\prod_{i=0}^{M-1} (z + \tilde{a}_i)} \frac{1}{\prod_{j=0}^{N-1} (z + \tilde{b}_j)}.
\]

(14)

where \( \tilde{a}_i = a_{Ri} + ja_{Mi} \) and \( \tilde{b}_j = b_{Rj} + jb_{Mj} \) are the zeros and poles, respectively. The transfer function of the required complex bandpass filter is then obtained by rotating \( \tilde{a}_i \) and \( \tilde{b}_j \) in the complex \( z \)-plane by equal angles \( \omega_c \) rad according to

\[
\begin{bmatrix}
\tilde{a}'_i \\
\tilde{a}_i
\end{bmatrix} = \begin{bmatrix}
\cos \omega_c & -\sin \omega_c \\
\sin \omega_c & \cos \omega_c
\end{bmatrix} \begin{bmatrix}
a_{Ri} \\
a_{Mi}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\tilde{b}'_j \\
\tilde{b}_j
\end{bmatrix} = \begin{bmatrix}
\cos \omega_c & -\sin \omega_c \\
\sin \omega_c & \cos \omega_c
\end{bmatrix} \begin{bmatrix}
b_{Rj} \\
b_{Mj}
\end{bmatrix},
\]

(15)

where \( \omega_c \) is the center frequency of the required complex bandpass filter \((-\pi \leq \omega_c < \pi)\) and \( \tilde{a}'_i = a'_{Ri} + ja'_{Mi} \), \( \tilde{b}'_j = b'_{Rj} + jb'_{Mj} \) are the rotated zeros and poles. The \( \tilde{a}'_i \) and \( \tilde{b}'_j \) from (15) are substituted into (14) to determine the complex coefficients \( \tilde{p}_i \) and \( \tilde{q}_i \) of the complex LDE in the form

\[
Y(\omega_1, \omega_2, n_3) = \tilde{a}_0^{\frac{1}{2}} \sum_{i=0}^{M-1} \tilde{p}_i X(\omega_1, \omega_2, n_3 - i) - \sum_{i=1}^{N-1} \tilde{q}_i Y(\omega_1, \omega_2, n_3 - i).
\]

(16)

Example: 3-D FP Filters using Real- Versus Complex-Coefficient Bandpass LDE Filters

A digitized video image sequence \( x(n_1, n_2, n_3) \) containing moving vehicles has been obtained and consists of 155 frames of dimension 128 \( \times \) 128. The 130th frame \( x(n_1, n_2, 130) \) is shown in Fig. 3. Two vehicles are shown traveling at the same velocity from the right side to the left side in the far lane of traffic along with a single vehicle traveling at approximately the same velocity, but opposite direction, in the near lane of traffic. A 3-D combined DFT/LDE FP filter has been designed to enhance the two vehicles in the far lane, using real-coefficient LDE's and having a passband plane \( P_1 \), given by \(-0.71 \omega_1 + 0.08 \omega_2 + 0 = 0\). The vehicle in the near lane, however, has most of its energy in the plane \( P_2 \), given by \(-0.71 \omega_1 + 0.08 \omega_2 - \omega_3 = 0\). Consequently, all three vehicles appear in the output \( y(n_1, n_2, 130) \), as shown in Fig. 4. A corresponding 3-D combined DFT/LDE FP filter has been designed with complex-coefficient LDE's. The complex LDE's are of order \( M = N = 2 \) and have been obtained from a second-order Butterworth low-pass prototype filter. The magnitude frequency response \( M(\omega_1, \omega_2, \omega_3) \) on \( \omega_3 \) and the coefficients \( \tilde{p}_i \) and \( \tilde{q}_i \) of one such complex filter are shown in Fig. 5 for the 2-tuple \((\omega_1, \omega_2) = (-\pi/2, -\pi/2)\). The passband is on plane \( P_1 \), with \( \omega_3 = -0.315 \pi \) and \( B = 0.0378 \pi \). The 130th frame \( y(n_1, n_2, 130) \) in the output of

![Fig. 2. Ideal Complex bandpass filter response.](image)

![Fig. 3. Single frame obtained from a digitized video image sequence containing moving vehicles.](image)

![Fig. 4. Filtered output using real-coefficient LDE's.](image)

![Fig. 5. Coefficients and magnitude response for complex filter used at 2-tuple \((-0.5 \pi, -0.5 \pi)\).](image)

![Fig. 6. Filtered output using complex-coefficient LDE's.](image)
the 3-D combined DFT/LDE with complex-coefficient bandpass LDE's is shown in Fig. 6, where it is observed that the time-reversed vehicle in the near lane has been successfully attenuated, due to the elimination of the passband plane \( P_T \).

IV. SUMMARY

Real-coefficient LDE's impose symmetries in \( H(j\omega_1, j\omega_2, j\omega_3) \) that leave just two independent octants in \( \Omega \). These symmetries imply that the 3-D combined DFT/LDE filter enhances signals and their time-reversed versions in the same way. This limitation may be overcome by employing complex-coefficient LDE filters, which allow four independent octants of \( H(j\omega_1, j\omega_2, j\omega_3) \). Conventional methods of 3-D-DFT filtering and 3-D-LDE filtering also have four independent octants in \( H(j\omega_1, j\omega_2, j\omega_3) \), where \( \omega_1, \omega_2, \omega_3 \in \Omega \) in the former case and \( \omega_1, \omega_2, \omega_3 \in \mathbb{R} \) in the latter case.

REFERENCES


On the Uncertainty Principle in Discrete Signals

Léon Claude Calvez and Pierre Vilbé

Abstract—It has recently been shown that the uncertainty principle holds true by appropriate definitions of the durations even if discrete signals are considered. A nice basic inequality was derived in the particular case where the Fourier transform is real. As an extension to this work, it is the purpose of this paper to prove the uncertainty relation in the general case of a complex Fourier transform and with somewhat extended definitions of durations.

The existence of basic limitations on the possibility of simultaneously confining the time and frequency spread of analog signals is known as the Uncertainty Principle, a term borrowed from quantum mechanics. The existence of similar limitations in discrete signals has recently been investigated by Ishii and Furukawa [1].

In this paper, for continuity and brevity, formulas and nomenclature follow directly from the development in [1]. As in [1], consider a band-limited analog signal \( f_x(t) \) whose Fourier transform \( F_x(\Omega) \) is

\[
F_x(\Omega) = 0, \quad \text{for} \quad |\Omega| > \sigma.
\]

Let a sequence \( f(n) = f_x(nT) \) be the sample values of \( f_x(t) \), where \( T = \pi/\sigma \). The Fourier transform and the inverse Fourier transform are expressed by the following equations:

\[
F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} f(n)e^{-j\omega n} \tag{1}
\]

\[
f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\omega})e^{j\omega n} d\omega \tag{2}
\]

where \( \omega = \Omega T \). The Fourier transforms are related by the relationship

\[
F(e^{j\omega}) = F_x(\Omega)/T, \quad |\omega| \leq \pi. \tag{3}
\]

Assuming that the energy of the signal is equal to 1, i.e.,

\[
\sum_{n=-\infty}^{\infty} |f(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{j\omega})|^2 d\omega = 1 \tag{4}
\]

we define the (time) spread \( D_n \) of \( f(n) \) by

\[
D_n = \sum_{n=-\infty}^{\infty} (n - n_0)^2 |f(n)|^2 \tag{5}
\]

and the (frequency) spread \( D_\Omega \) of \( F(e^{j\omega}) \) by

\[
D_\Omega = \int_{-\pi}^{\pi} (\omega - \omega_0)^2 |F(e^{j\omega})|^2 d\omega \tag{6}
\]

where \( n_0 \) and \( \omega_0 \) are real numbers. It is worth noting that several analytical definitions of duration have been suggested in the past for analog signals and recently for discrete signals [1]; for a convenient choice of \( n_0 \) and \( \omega_0 \), within a multiplicative constant, the above definitions of \( D_n \) and \( D_\Omega \) can be given an interesting interpretation [2]. If \( n_0 = \omega_0 = 0 \), they reduce to those of [1]. It is a standard exercise to show that if \( |f(n)| \) is an even function of \( n \), then \( n_0 = 0 \) minimizes (5), and that when \( f(n) \) is real, then \( |F(e^{j\omega})|^2 \) is an even function of \( \omega \) and \( \omega_0 = 0 \) minimizes (6).

In the particular case where \( F(e^{j\omega}) \) is real (a serious restriction), Ishii and Furukawa [1] have shown that the nice inequality

\[
D_nD_\Omega > \frac{\pi}{2} \tag{7}
\]

holds for \( n_0 = \omega_0 = 0 \) provided that \( F_x(\Omega) = 0 \) for \( |\Omega| = \sigma \). As an extension to this work we shall prove that the uncertainty relation (7) holds for any complex Fourier transform and for any choice of \( n_0 \) and \( \omega_0 \), provided that \( F_x(\Omega) = 0 \) for \( |\Omega| = \sigma \). This can be proved as follows.

Consider the integral

\[
I = \int_{-\pi}^{\pi} \Phi(\omega)\Phi^*(\omega) d\omega \tag{8}
\]

where superscript \( * \) stands for complex conjugate and

\[
\Phi(\omega) = e^{j\omega_0}\Phi(e^{j\omega}) \tag{9}
\]

\[
\Phi^*(\omega) = \frac{d\Phi(\omega)}{d\omega}. \tag{10}
\]

Integration by parts yields

\[
I = K - \int_{-\pi}^{\pi} \Phi(\omega)^2 \Phi^*(\omega) d\omega \tag{11}
\]

with

\[
K = \left[ (\omega - \omega_0)|F(e^{j\omega})|^2 \right]_{-\pi}^{\pi} - \omega_0\left| F(e^{j\omega}) \right|^2 \tag{12}
\]

and

\[
K = \pi \left[ |F(e^{j\omega})|^2 + |F(e^{-\omega})|^2 \right] \tag{13}
\]

where

\[
\theta = \pi - \left[ |F(e^{j\omega})|^2 - |F(e^{-\omega})|^2 \right]. \tag{14}
\]