of the deconvolution algorithm, the input signal $x(n)$ is computed with no error.

V. DISCUSSION

The new computational algorithm works for any kind of deconvolution kernel—nonsingular or singular. It is much faster than the time-domain method and its complexity can be compared to that of the FFT. In contrast to the direct DFT deconvolution, this method uses only multiplications and no division except for the final scaling.

We also tested the described method in a noisy environment. The noise behavior of the algorithm was tested by adding zero-mean white noise with preselected signal-to-noise ratios (SNR's) to the output signal sequences and to the kernels, respectively. The experiments were carried out on real signals; we took them from the American Heart Association Database of standard electrocardiograms. Kernels $h(n)$ were obtained as the averages of 500 ECG periods and truncated to length $N/2$. Randomly chosen extrastoles from the same ECG were considered the output signals $y(n)$ of length $N(N = 256)$. To achieve statistical significance, 15 different ECG's, two leads from each, were tested and each measurement was averaged across 20 runs.

We investigated input SNR's (denoted by SNR,) versus output SNR's (SNR,) and versus kernels' SNR's (SNR,) as explained by Fig. 6. Input SNR's are presented with their average (circles) and the double standard deviation (±σ). The solid lines correspond to deconvolution without division (the described method), the dashed lines to the direct deconvolution with interpolation. We made the kernels singular synthetically by replacing with zeros all the frequency samples $H(k)$ whose magnitudes were lower than 1.5 times the smallest magnitude value.

As it was shown in [9], deconvolution with interpolation performs 3 dB better on average than the direct DFT deconvolution. According to noise study here, the achieved SNR's are practically the same for deconvolution with interpolated samples when the output is noisy (Fig. 6(a)), while the results are 0.7 dB better in favor of the presented method when noise is added to the deconvolution kernels (Fig. 6(b)).

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The Two-Dimensional Complex LMS Algorithm Applied to the 2-D DFT

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Abstract—The one-dimensional (1-D) LMS spectrum analyzer, as proposed by Widrow, is extended to the two-dimensional (2-D) case. The relationship between the 2-D discrete Fourier transform (DFT) and the 2-D least mean square (LMS) algorithm is established and employed to recursively compute a 2-D LMS spectrum, or the 2-D DFT, by proper choice of the LMS adaptation step $\mu$. The Two-Dimensional Complex LMS Algorithm Applied to the 2-D DFT

1. INTRODUCTION

Widrow et al. have shown [1] that the complex LMS adaptive algorithm may be employed to recursively compute the one-dimensional (1-D) discrete Fourier transform (DFT) of a real or complex sequence $d_{k}$, and to obtain a 1-D spectral estimate of $d_{k}$. This method employs the conventional 1-D adaptive linear combiner, having the adaptive weight column vector $W_{k} \equiv \begin{bmatrix} w_{k_{1}}, w_{k_{2}}, \ldots, w_{k_{N_{n}}-1} \end{bmatrix}^{T}$ and the input column vector $X_{k} \equiv \begin{bmatrix} x_{k_{1}}^{(1)}, x_{k_{1}}^{(2)}, \ldots, x_{k_{N_{n}}-1}^{(2)} \end{bmatrix}^{T}$, to generate the adaptive signal estimate $y_{k} \equiv W_{k}^{T}X_{k}$. The essential feature of Widrow’s method is that, for the computation of the DFT of a signal $d_{k}$, of block length $N$, the input column vector $X_{k}$ is the set of $N$ periodic discrete phasor sequences given by $X_{k} = \begin{bmatrix} 1/\sqrt{N} \cdot e^{j2\pi N_{1}/N}, \ldots, e^{j2\pi N_{N_{n}-1}/N} \end{bmatrix}^{T}$ with $\mu \equiv \sqrt{-1}$, and the mean squared value of the complex error $e = d_{k} - y_{k}$ is minimized according to the complex LMS algorithm. Widrow et al. showed that the algorithm operates as a spectrum analyzer and the output vector of this spectrum analyzer is then a weighted spectral estimate of the sequence $d_{k}$. The weighting is controlled by the LMS adaptation (or convergence) step $\mu$, and for $\mu \equiv 1/2$, this output vector is exactly the DFT of $d_{k}$, over the block length of $N$ defined by the index range $k_{1} \cdot N$ to $k_{N_{n}} - 1$.

In Section II of this paper, the 2-D complex LMS algorithm is developed and applied to a sequence of 2-D images. In Section III, the 2-D DFT spectrum analyzer is proposed for obtaining the 2-D DFT of images or the “steady-flow” 2-D DFT of sequences of 2-D images. The algorithm is summarized in Fig. 2, where the 2-D exponential basis functions $(1/\sqrt{N_{1}N_{2}}) \cdot e^{j2\pi(N_{1}a_{1}+N_{2}a_{2})}$ at

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the outputs of the multipliers are adaptively weighted by the 2-D weights \([w_{r,q}]\). The (desired) input signal \(x(u_1,u_2)\) is the sequence of 2-D images and the \(N_x \times N_y\) output matrix \([u_{r,q}]\) is then the required 2-D DFT.

II. THE PROPOSED 2-D COMPLEX LMS ALGORITHM

The infinite sequence of \(N \times N\) 2-D images in Fig. 1 may be represented as an infinite length 2-D strip of width \(N_y\), as shown. This 2-D strip \(x(u_1,u_2)\) has region of support \(R_x = \{u_1,u_2\} 0 \leq u_1 \leq N_x, 0 \leq u_2 \leq (N_y - 1)\). \(N_x < \infty\) and, as shown in Fig. 1, the variables \(n_r\) and \(n_g\) may be used to point to the data sample \(x(u_1,u_2)\) in row \(n_r\) of frame \(n_f\), where \(n_1 = n_rN_x + n_g\). This infinite strip \(R_x\) has frame boundaries at integer multiples of \(N_x\).

The direction of recursion for the 2-D LMS algorithm is assumed to be pixel-by-pixel in Fig. 1, starting at \((u_1,u_2) = (0,0)\) and proceeding first along row 0, then row 1, etc. to row \((N_y-1)\), in order to coincide with the order of transmission of the conventional signal. The final computation in frame \(n_f = 0\) is therefore at \((u_1,u_2) = (N_x-1,N_y-1)\), after which the next computation is at \((u_1,u_2) = (N_x,0)\) and is the first computation in frame \(n_f = 1\). With this notation and ordering of computations, the application of the 2-D LMS algorithm to image sequences is an algebraically consistent extension of the 1-D case [1].

2.1. The 2-D Adaptive Linear Combiner

Assume that a 2-D adaptive linear combiner has the input signal \(U(u_1,u_2) = [u_{r,q}(u_1,u_2)]\) and the weight matrix \(W(u_1,u_2) = [w_{r,q}(u_1,u_2)]\), where \(u_{r,q}(u_1,u_2)\) and \(w_{r,q}(u_1,u_2)\) respectively denote the elements in the \(r\)th row and \(q\)th column of \(U(u_1,u_2)\) and \(W(u_1,u_2)\), and \(r = 0,1,\ldots,N_x\) and \(q = 0,1,\ldots,N_y\). The output signal \(\hat{x}(u_1,u_2)\) of the adaptive linear combiner is then

\[
\hat{x}(u_1,u_2) = \sum_{r=0}^{N_x-1} \sum_{q=0}^{N_y-1} u_{r,q}(u_1,u_2) w_{r,q}(u_1,u_2) \tag{2.1}
\]

and the error signal \(e(u_1,u_2)\) required for adaptation is defined as

\[
e(u_1,u_2) = x(u_1,u_2) - \hat{x}(u_1,u_2). \tag{2.2}
\]

2.2 The 2-D Complex LMS Algorithm Applied to a Sequence of 2-D Images

The general idea of the proposed 2-D complex LMS algorithm is to adapt the weight matrix \(W(u_1,u_2)\) as the recursion proceeds to minimize the mean squared error using a similar strategy to the 1-D complex LMS algorithm [2]. To derive the LMS algorithm, the gradient of the mean squared error is estimated by the instantaneous gradient of the squared error at each iteration and the method of steepest descent is then applied [3]. It is easy to observe that the complex LMS algorithm should be able to adapt the real and imaginary parts of \(W(u_1,u_2)\) simultaneously to minimize both the real and imaginary parts of \(e(u_1,u_2)\) in the least mean square sense.

Let \(W(u_1,u_2)\) be expressed in terms of its real and imaginary parts, thus

\[
W(u_1,u_2) \equiv W_r(u_1,u_2) + jW_i(u_1,u_2). \tag{2.3}
\]

Following the conventional LMS optimization strategy, we use the method of steepest descent applied separately to the real and imaginary parts of the weight matrix \(W(u_1,u_2)\) to determine the update equations as follows.

\[
W'_r(u_1,u_2) = W_r(u_1,u_2) - \mu \nabla W_r \|e(u_1,u_2)\|^2 \tag{2.4a}
\]

and

\[
W'_i(u_1,u_2) = W_i(u_1,u_2) - \mu \nabla W_i \|e(u_1,u_2)\|^2 \tag{2.4b}
\]

where \(W'\) represents the weight matrix after updating and \(\mu\) is the adaptation step, so that

\[
W' = W(u_1,u_2) - \mu [\nabla W_r \|e(u_1,u_2)\|^2] + j \nabla W_i \|e(u_1,u_2)\|^2. \tag{2.5}
\]

Consider a complex-valued function \(f = f(x,y)\), then \(|f|^2 = \bar{f} f\) so that

\[
\frac{\partial |f|^2}{\partial x} = f \frac{\partial \bar{f}}{\partial x} + f \frac{\partial f}{\partial x}, \quad \text{and} \quad \frac{\partial |f|^2}{\partial y} = f \frac{\partial \bar{f}}{\partial y} + f \frac{\partial f}{\partial y}. \tag{2.6}
\]

With \(f = \gamma(x + jy)\), \(\gamma\) a complex constant, we obtain

\[
\frac{\partial f}{\partial x} = \gamma, \quad \text{and} \quad \frac{\partial f}{\partial y} = j \gamma. \tag{2.7}
\]
so that
\[
\frac{\partial |f|^2}{\partial x} + j \frac{\partial |f|^2}{\partial y} = 2\gamma f.
\]  
(2.8)

Applying this result as well as (2.1) and (2.2) to the second term on the right side of (2.5) yields the 2-D complex LMS algorithm as
\[
W^\alpha = W(n_1, n_2) + 2\mu \sum_{n_1} W(n_1, n_2) \tilde{U}(n_1, n_2).
\]  
(2.9)

This algorithm can be expressed more specifically for the chosen recursion strategy as
\[
W(n_1, n_2 + 1) = W(n_1, n_2) + 2\mu \Xi(n_1, n_2) \tilde{U}(n_1, n_2),
\]
\[
0 \leq n_1 < +\infty, 0 \leq n_2 < N_2 - 1
\]  
(2.10a)

and
\[
W(n_1 + 1, n_2) = W(n_1, n_2 - 1) + 2\mu \Xi(n_1, n_2) \tilde{U}(n_1, n_2 - 1),
\]
\[
0 \leq n_1 < +\infty.
\]  
(2.10b)

(This result may be generalized to the M-dimensional case \( M > 2 \), but is pursued here only for the 2-D case). In the following section, it is shown that this 2-D complex LMS algorithm may be used to compute the weighted 2-D spectrum of a sequence of images and also the 2-D DFT of a single image frame.

III. THE RELATIONSHIP BETWEEN THE 2-D DFT AND THE 2-D COMPLEX LMS ALGORITHM

The relationship between (2.10) and the 2-D DFT of image sequences is now developed. It is now shown that the above 2-D Complex LMS Algorithm may be used to implement a 2-D LMS Spectrum Analyzer which, for a particular choice of \( \mu \), leads to the exact computation of the 2-D DFT.

3.1. The 2D LMS Spectrum Analyzer

The proposed 2-D spectrum analyzer is shown in Fig. 2, where the input signal of the 2-D adaptive linear combiner is a \( N_1 \times N_2 \) 2-D complex phasor sequence array given by
\[
U(n_1, n_2) = P_1(n_1) P_2^T(n_2)
\]  
(3.1)

where
\[
P_1(n_1) = \frac{1}{\sqrt{N_1}} \left[ 1, e^{j\frac{2\pi}{N_1} n_1}, \ldots, e^{j\frac{2\pi(n_1-1)}{N_1} n_1} \right]^T
\]

and
\[
P_2(n_2) = \frac{1}{\sqrt{N_2}} \left[ 1, e^{j\frac{2\pi}{N_2} n_2}, \ldots, e^{j\frac{2\pi(n_2-1)}{N_2} n_2} \right]^T
\]

We define the trailing block with respect to \( (n_1, n_2) \) as the \( N_1 \times N_2 \) sample points that trail the sample point \( (n_1, n_2) \) in the recursion as shown shaded in Fig. 3(a). It will be useful to consider the trailing block \( X_{TB}(n_1, n_2) \) in the rectangular form \( X_{TB}(n_1, n_2) \) shown in Fig. 3(b). Clearly the rectangular form of \( X_{TB}(n_1, n_2) \) does not then correspond to an image frame except where \( (n_1, n_2) = (n_1, N_2 - 1) \), in which case, \( X_{TB}(n, N_2, 0) \) corresponds to frame number \( (n, -1) \) in the original sequence.
a 1-D algorithm:
\[
V(i+1) = V(i) + 2\mu \chi(i) \hat{p}(i) \tag{3.5}
\]
or
\[
V(i+1) = (I - 2\mu \hat{p}(i) \hat{p}^T(i))V(i) + 2\mu \chi(i) \hat{p}(i) \tag{3.6}
\]
where \(\chi(i) = x(n_1, n_2)\), and
\[
\hat{\chi}(i) = z(n_1, n_2) = \chi(i) - V^T(i)\hat{p}(i). \tag{3.7}
\]
Noting that \(\rho^T(k)\hat{p}(m) = 1\) if \(k = m + lM\) for some integer \(l\) and 0 otherwise, and assuming \(V(0) = 0\), we obtain
\[
V(i) = 2\mu \sum_{l=-\infty}^{-1} (1 - 2\mu)^l \sum_{m=-l}^{-1} \chi(m) \hat{p}(m) \tag{3.8}
\]
where we have assumed \(\chi(m) = 0\) for \(m < 0\). Note that if \(\mu = 1/2\), then all but the \(l = 0\) term are zero. Letting \(\mu = 1/2\), and \(m = m_1 N_2 + m_2\), we find
\[
W(n_1, n_2) = \sum_{m_1=-l}^{n_1-1} \frac{\sum_{m_2=0}^{n_2-1} Q(m_1, m_2)}{\frac{N_1}{2} \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{N_2-1} Q(m_1, m_2)} \sum_{m_1=-l}^{n_1-1} \frac{Q(m_1, m_2)}{\frac{N_1}{2} \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{N_2-1} Q(m_1, m_2)} \tag{3.9}
\]
where \(Q(m_1, m_2) = x(m_1, m_2)P_1^T(m_1)P_2^T(m_2)\). From (3.2) and (3.9), it follows that
\[
O(n_1, n_2) = \sum_{n_1=-l}^{n_1-1} \frac{\sum_{n_2=-l}^{n_2-1} g(m, n)P_1^T(m)P_2^T(n)}{\frac{N_1}{2} \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{N_2-1} Q(m_1, m_2)} \tag{3.10}
\]
where
\[
g(m, n) = \begin{cases} 
\frac{x(m + n_1 - N_1, n + n_2 - N_2)}{N_1 - 1}, & 1 \leq m \leq N_1 - 1, \\
\frac{x(m + n_1 - N_1, n + n_2)}{N_1 - 1}, & 1 \leq m \leq N_1 - 1, \\
0, & 0 \leq m \leq N_1 - 1, \\
\frac{x(m + n_1 - N_1, n + n_2 - 1)}{N_1 - 1}, & 0 \leq m \leq N_1 - 1, \\
0, & 0 \leq n \leq N_2 - n_2 - 1, \\
x(m_1, n + n_2 - N_2), & m = 0, \\
x(m_1, n + n_2), & N_2 - n_2 \leq n \leq N_2 - 1.
\end{cases} \tag{3.11a-3.11c}
\]
Consequently, \(O(n_1, n_2)\) is exactly the 2-D DFT of \(X_{TH}(n_1, n_2) = [p(m, n)]\). If \((n_1, n_2) = (n_N, N_2)\), then \(g(m, n) = x(m + n_1 - 1, N_1, n)\) and \(X_{TH}(n_1, n_2) = X_{TH}(n_1, n_2)\). We then have, in general,
\[
O(n_1, n_2) = 2 \text{DFFT}[X_{TH}(n_1, n_2)]. \tag{3.12}
\]
That is, with \(\mu = 1/2\), the output matrix \(O(n_1, n_2)\) at points \((n_N, N_2)\) is exactly the 2-D DFT of frame number \((n_F - 1)\). Therefore, this method may be used to compute the 2-D DFT of the 2-D frames in an infinite sequence of frames. The output matrix \(O(n_1, n_2)\) of the 2-D LMS spectrum analyzer is then the required 2-D DFT.

3.2. The “Steady-flow” 2-D DFT
It is shown from (3.10) and (3.11) that \(O(n_1, n_2)\), for \(\mu = 1/2\), is the 2-D DFT of \(X_{TH}(n_1, n_2) = [g(m, n)]\). Comparing (3.11) with the diagrammatic partitioning of the trailing block in Fig. 3, (3.11a) corresponds to subblock A, (3.11b) to subblock B, and (3.11c) to subblock C. A. Hence, we call this kind of 2-D DFT the “steady-flow” 2-D DFT. The output of the spectrum analyzer \(O(n_1, n_2)\) is equal to the “steady-flow” 2-D DFT defined above at any point \((n_1, n_2)\).
It represents the steady change of the 2-D spectrum of the sequence of images. It can be used in the tracking of dynamic objects in image sequence and 2-D frequency-domain adaptive filters [6].

By using the relations in (3.2) and (3.8), the output \( O(n_1, n_2) \) of the "steady-flow" 2-D DFT spectrum analyzer can be written as the geometric sum:

\[
O(n_1, n_2) = 2\mu \sum_{l=0}^{\infty} \left(1 - 2\mu^2\right)^l \text{DDFT} [X_{\text{HT}}(n_1, n_2)].
\]

(3.13)

This geometric coherent-average "steady-flow" 2-D DFT, a new form of the DFT, is evaluated from (3.13) by choosing \( \mu \) as an appropriate "memory factor" of the 2-D DFT's of previous frames. The stability of the complex 2-D LMS/DFT algorithm is guaranteed if \( |1 - 2\mu| < 1 \), requiring that \( 0 < \mu < 1 \). Larger deviations from the value of \( \mu = 1/2 \) in this interval imply a long "memory factor" and, at \( \mu = 1/2 \), the "memory factor" is zero according to (3.13).

3.3 Numerical Verification

Consider the simple test signal

\[
x(n_1, n_2) = 128 \left( \sin \left( \frac{2\pi (5n_1 + 5n_2)}{32} \right) + \cos \left( \frac{2\pi (10n_1 + 10n_2)}{32} \right) \right)
\]

(3.14)

where the image size is given by \((N_1, N_2) \equiv (32, 32)\).

The magnitude of the 2-D LMS/DFT of \(x(n_1, n_2)\), computed according to the algorithm proposed in this contribution, is shown in Fig. 4 and corresponds with that obtained using the conventional 2-D DFT method of calculation.

3.4. Remarks on Computational Complexity

For complex data \(x(n_1, n_2)\), the conventional 2-D FFT of the \(N_1 \times N_2 \) image \(x_{\text{HT}}(n_1, n_2)\) requires \(2N_1N_2 \log_2(N_1N_2)\) real multiplications and \(3N_1N_2 \log_2(N_1N_2)\) real additions, whereas it can easily be shown that the 2-D LMS/DFT algorithm proposed in this paper requires \(8N_1N_2\) real multiplications and \(10N_1N_2 - 4N_1 - 4N_2 + 6\) real additions for \(X_{\text{HT}}(n_1, n_2)\). Therefore, for \(N_1, N_2 \gg 1\), the 2-D LMS/DFT algorithm is significantly more efficient in trailing block applications, such as adaptive filtering where the trailing block concept ensures smooth variations in the adaptive weights retained over translations between very different image frames. Of course, for calculating only the 2-D DFT of separate image frames, the 2-D LMS/DFT algorithm is significantly less efficient because the recursion must proceed through \(N_1N_2\) steps in \((n_1, n_2)\) between frame boundaries. It can be shown, however, that by updating the 2-D complex LMS algorithm on a row-by-row (rather than pixel-by-pixel) basis, this 2-D LMS/DFT may be vectorized and thereby the processing speed increased by the factor \(N_2 - 1\) [6].

IV. CONCLUSIONS

In this contribution, Widrow's 1-D spectrum analyzer has been extended to the 2-D case to establish the relationship between the 2-D DFT and the 2-D complex LMS algorithm. It results directly in the 2-D LMS spectrum analyzer, which is a new recursive method to calculate the 2-D DFT. Using the conventional frame-by-frame computations of the 2-D DFT's to compute the geometric coherent-average "steady-flow" 2-D DFT (3.13) would require that the 2-D DFT's of many previous frames to be stored in memory whereas the proposed method requires no memory storage of previous image samples because it is totally recursive. However, internally, the algorithm does require that \(W(n_1, n_2), P_1(n_1), \) and \(P_2(n_2)\) be stored. The 2-D LMS algorithm has the advantage that it allows concurrent computations of the \(N_1 \times N_2\) terms in the update matrix equation (2.10) and therefore the potential for very fast parallel computations of the 2-D DFT.

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