The Uniqueness in Designing Multidimensional Causal Recursive Digital Filters Possessing Magnitude Hyperspherical Symmetry

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***Abstract*—It is shown that magnitude hyperspherically symmetric transfer functions of multidimensional (MD) causal recursive digital filters must have numerator and denominator polynomials that are separately magnitude hyperspherically symmetric. Further, the exact reference-domain magnitude-hyperspherically symmetric denominator polynomial is of infinite order, possessing only one free parameter, and the magnitude hyperspherically symmetric numerator polynomial *itself* has to be a radial even function. The corresponding MD design problem is shown to be essentially a one-dimensional design problem. Filter transfer functions having good symmetry and moderate degree can be designed by using the presented procedure.**

I. Introduction

The design of two-dimensional (2-D) circularly symmetric and 3-D spherically symmetric filters has been studied extensively [1]–[3] and a number of useful numerical design procedures have emerged [4]–[6]. However, closed analytic solutions for such symmetrical filters do not so far exist and the extensions to the more general case of n-dimensional (n-D) hyperspherical symmetry ($n > 3$) have not been established.

In general, numerical design methods benefit significantly from *a priori* knowledge of explicit relationships between the coefficients of the numerator as well as denominator polynomials; for example, by allowing the number of variables of optimization to be reduced and by choosing suitable initial values of these variables. In this contribution, such explicit relations on the coefficients are derived for n-D hyperspherically symmetric causal recursive filter transfer functions.

The transfer function of an n-D recursive digital filter, say $H_d(z)$, is a rational function of the variables z_i ($i = 1$ to n) with

$$z_i = e^{s_i T_i}, \tag{1}$$

where $z = (z_1, z_2, \cdots, z_n)^T$ (the superscript T denotes the transpose), T_i is a positive constant and

$$s_i = \sigma_i + j\omega_i$$

is the complex frequency in the i-th frequency dimension. Thus, the frequency dependence of the transfer function is not rational. In order to obtain a rational transfer function in the so-called *equivalent* frequency variables, which is more plausible and familiar, the bilinear transformation

$$\psi_i = \frac{z_i - 1}{z_i + 1} = \tanh \frac{s_i T_i}{2},$$

is applied to the z-domain transfer function $H_d(z)$. The obtained bilinear transformed transfer function (or the $\boldsymbol{\psi}$-domain transfer function with $\boldsymbol{\psi} = (\psi_1, \psi_2, \cdots, \psi_n)^T$), say

$$H(\boldsymbol{\psi}) = H_d(z)|_{z_i = (1+\psi_i)/(1-\psi_i)},$$

is then a rational function in the variables ψ_i ($i = 1$ to n) with

$$\psi_i = \delta_i + j\varphi_i,$$

which is called the *complex reference frequency* (or the *complex equivalent frequency*) in the i-th frequency dimension. The relation between the real frequency ω_i and the real reference frequency φ_i ($i = 1$ to n) is given by

$$\varphi_i = \tan \frac{\omega_i T_i}{2}. \tag{2}$$

Given the n-D z-domain transfer function $H_d(z)$ and the corresponding $\boldsymbol{\psi}$-domain transfer function $H(\boldsymbol{\psi})$, the *squared magnitude frequency responses* in the corresponding real frequency domains $\boldsymbol{\omega}$ and $\boldsymbol{\varphi}$ ($\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_n)^T$ and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_n)^T$) are defined by

$$M_d^2(\boldsymbol{\omega}) = H_d(z)H_d(z^{-1})|_{z_i = e^{j\omega_i T_i}\,(i = 1 \text{ to } n)}$$

and

$$M^2(\boldsymbol{\varphi}) = H(j\boldsymbol{\varphi})H(-j\boldsymbol{\varphi}),$$

where $z^{-1} = (z_1^{-1}, z_2^{-1}, \cdots, z_n^{-1})^T$. We are concerned here about the symmetries of such functions. In particular, we say *magnitude symmetry* (MS) for the $\boldsymbol{\omega}$-domain or $\boldsymbol{\varphi}$-domain symmetries possessed by the (squared) magnitude responses $M_d^2(\boldsymbol{\omega})$ or $M^2(\boldsymbol{\varphi})$ and *polynomial symmetry* (PS) for the $\boldsymbol{\omega}$-domain or $\boldsymbol{\varphi}$-domain symmetries possessed by the numerator and denominator polynomials of the

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frequency responses $H_d(z)$ ($z_i = e^{j\omega_i T_i}$, $i = 1$ to n) or $H(j\boldsymbol{\varphi})$ themselves. The relationship of symmetries in the $\boldsymbol{\omega}$-domain and $\boldsymbol{\varphi}$-domain is determined by the well-known mapping properties (2). Many symmetries, such as hyperquadrantal symmetry, are preserved over this mapping. Hence, in most cases, it is sufficient only to consider the rational transfer function $H(\boldsymbol{\psi})$ in the reference frequency domain $\boldsymbol{\psi}$. The reference domain transfer function $H(\boldsymbol{\psi})$ can be considered as the transfer function of a corresponding classical analog filter [7], [8].

In the following, we shall be concerned with the magnitude hyperspherical symmetry. However, for easy reference, the properties of two related symmetries, that are necessary for the magnitude hyperspherical symmetry, namely, magnitude hyperquadrantal and magnitude hyperoctagonal symmetry, are briefly discussed.

A. Magnitude Hyperquadrantal and Hyperoctagonal Symmetry (MHQS, MHOS)

Definition 1: An n-D filter transfer function $H_d(z)$ is *magnitude hyperquadrantally symmetric* (MHQS) in the $\boldsymbol{\omega}$-domain, if the value of its (squared) magnitude response $M_d^2(\boldsymbol{\omega})$ is not altered by replacing any of the z_i, $i = 1$ to n, with the respective z_i^{-1} in $H_d(z)$, or equivalently, any of the ω_i with the $-\omega_i$ in $M_d^2(\boldsymbol{\omega})$.

Definition 2: An n-D filter transfer function $H_d(z)$ is *magnitude hyperoctagonally symmetric* (MHOS) in the $\boldsymbol{\omega}$-domain, if it is magnitude hyperquadrantally symmetric and if the value of its (squared) magnitude response $M_d^2(\boldsymbol{\omega})$ is not altered by arbitrarily exchanging the positions of any of the z_i ($i = 1$ to n) in the transfer function $H_d(z)$ with any other z_j ($j = 1$ to n) in $H_d(z)$, or equivalently, all the frequency axes ω_i ($i = 1$ to n) are equivalent.

Correspondingly, we define $\boldsymbol{\varphi}$-domain magnitude hyperquadrantal and hyperoctagonally symmetry. According to (2), it is obvious that MHQS and MHOS in the $\boldsymbol{\omega}$-domain involve the same symmetry in the $\boldsymbol{\varphi}$-domain and vice versa.

In the 2-D and 3-D cases, MHQS is referred to as magnitude quadrantal symmetry and magnitude cubic symmetry, respectively [11], [12], and the 2-D MHOS is referred to as magnitude octagonal symmetry [15].

It was first shown in [11] that the 2-D allpass-free MHQS transfer function having very strict Hurwitz denominator polynomial possesses 2-D MHQS numerator and 2-D MHQS separable denominator. This result has been generalized [13] to the MD case and is restated below:

Theorem 1: An n-D allpass-free reference domain transfer function $H(\boldsymbol{\psi})$, for which

$$H(\boldsymbol{\psi}) = \frac{f(\boldsymbol{\psi})}{g(\boldsymbol{\psi})}, \tag{3}$$

where $f(\boldsymbol{\psi})$ and $g(\boldsymbol{\psi})$ are co-prime (or irreducible) polynomials and $g(\boldsymbol{\psi})$ is furthermore a scattering Hurwitz polynomial [14], is magnitude hyperquadrantally symmetric if and only if the squared magnitudes of the numerator and denominator, $f(j\boldsymbol{\varphi})f(-j\boldsymbol{\varphi})$ and $g(j\boldsymbol{\varphi})g(-j\boldsymbol{\varphi})$, are individually hyperquadrantally symmetric and $g(\boldsymbol{\psi})$ is separable in its individual frequency variables, implying that

$$g(\boldsymbol{\psi}) = \gamma \prod_{i=1}^{n} g_i(\psi_i),$$

where γ is a unimodular constant.

Obviously, this theorem also holds for the corresponding z-domain transfer function, according to the equivalence statement on the MHQS in the $\boldsymbol{\omega}$- and the $\boldsymbol{\varphi}$-domain below Definition 2.

B. Magnitude Hyperspherical Symmetry (MHSS)

Definition 3: An n-D filter transfer function $H_d(z)$ is *magnitude hyperspherically (or radially) symmetric* (MHSS) in the $\boldsymbol{\omega}$-domain, if the value of its (squared) magnitude response $M_d^2(\boldsymbol{\omega})$ at any frequency point $\boldsymbol{\omega}$ in the n-D $\boldsymbol{\omega}$-space only depends on the Euclidian norm $\|\boldsymbol{\omega}\|$ with $\|\boldsymbol{\omega}\| = \sqrt{\omega_1^2 + \omega_2^2 + \cdots + \omega_n^2}$.

Correspondingly, we define $\boldsymbol{\varphi}$-domain magnitude hyperspherical symmetry. It is evident that a transfer function $H(\boldsymbol{\psi})$, that is, MHSS in, say, the $\boldsymbol{\varphi}$-domain does not retain the MHSS property under the mapping (2). Further, it is shown here that a z-domain transfer function $H_d(z)$ cannot be MHSS in the $\boldsymbol{\omega}$-domain. However, it is often the case that, if $H(\boldsymbol{\psi})$ is MHSS in the $\boldsymbol{\varphi}$-domain, then the bilinear transformed version $H_d(z)$ is approximately MHSS in the $\boldsymbol{\omega}$-domain in the region given by

$$\omega_i T_i \ll \pi$$

implying

$$\varphi_i \approx \omega_i T_i / 2, \qquad i = 1 \text{ to } n.$$

This approximation, used in combination with other frequency transformation techniques [9], [10] (cf. Section IV), offers very good symmetry properties. This is sufficient for most practical applications because the transition band, where symmetry is usually critically important, is rarely located near the Nyquist boundary π.

In the 2-D and 3-D cases, MHSS is referred to as magnitude circular symmetry and magnitude spherical symmetry, respectively.

C. Design Implications

Investigations on the properties of polynomials (which are simply special cases of transfer functions) having magnitude quadrantal and octagonal as well as cubic symmetry have been made in [11], [12], [15], where the classes of such symmetric polynomials are identified. This contribution establishes the necessary and sufficient conditions on the coefficients of the MD numerator and denominator polynomials having magnitude hyperspherical symmetry (MHSS). Indeed, we give these conditions in an explicit form in Sections II and III and show that they may be used to significantly simplify the filter design problem; for example, by increasing the probability of obtaining a nearly-optimal design during numerical optimization. A simplified formula for the magnitude hyperspherically symmetric polynomials is presented which can simplify the analysis of the filter transfer function and allow us to find an empirical design procedure, as shown in Section IV.

The requirements of MHQS and MHOS are obviously necessary conditions for MHSS transfer functions, and according to Theorem 1, an MHSS transfer function must then have a denominator polynomial that is separable in all the frequency variables. Intuitively, one might expect that it will be difficult to achieve an MHSS denominator polynomial, that is known, together with the MHSS numerator polynomial, as a *sufficient* condition for obtaining MHSS transfer functions. A *sufficient* condition for such $\boldsymbol{\psi}$-domain denominator polynomials being MHSS is given in [16], where the $\boldsymbol{\varphi}$-domain squared magnitude of the denominator, say $G(\boldsymbol{\varphi}^2)$ ($\boldsymbol{\varphi}^2 = (\varphi_1^2, \varphi_2^2, \cdots, \varphi_n^2)^T$), may be chosen in the form

$$G(\boldsymbol{\varphi}^2) = \prod_{i=1}^{n} G_0(\varphi_i^2), \tag{4a}$$

where

$$G_0(\varphi_i^2) \approx e^{\alpha \varphi_i^2}. \tag{4b}$$

We show in Section II that this is also a *necessary condition*, within a multiplicative constant. This new result is very restrictive and has the consequence that *the denominator polynomial of any exactly MHSS transfer function is of infinite order and fully determined except for the α parameter in* (4b). From the point of view of the filter design, the finite-order denominator polynomial must be determined to approximate (4b) as close as possible, in order to obtain a nearly-optimal solution. It has been shown [16], [17] that finite-order MHSS filter transfer functions satisfying the condition (4) can be obtained, which exhibit good selectivity and symmetry. In Section V, we show that the filter degree can be significantly reduced by using only a 1-D optimization procedure.

Although many symmetry considerations can be generalized to the case of *complex* transfer functions [13], [18], we focus here on *real* transfer functions.

II. Necessary and Sufficient Conditions on the Denominator $g(\psi)$ of MD Magnitude Hyperspherically Symmetric Transfer Functions

MD MHSS transfer functions are necessarily MHQS and MHOS. Therefore, we first briefly discuss the necessary and sufficient conditions on the denominator $g(\boldsymbol{\psi})$ of MD MHQS and MHOS transfer functions, in order to derive the corresponding conditions on the denominator $g(\boldsymbol{\psi})$ of MHSS transfer functions.

A. Magnitude Hyperquadrantal and Hyperoctagonal Symmetry (MHQS, MHOS) of $g(\boldsymbol{\psi})$

It follows from Theorem 1 that it is necessary and sufficient that the denominator $g(\boldsymbol{\psi})$ of an MHQS transfer function is a product of n 1-D (strict) Hurwitz polynomials. Hereafter, we first extend Theorem 1 to MHOS symmetry and show that a similar result holds for MHOS symmetry. A similar extension is also made by Theorem 3 for MHSS symmetry in the next subsection.

Theorem 2: An n-D allpass-free and irreducible reference domain transfer function $H(\boldsymbol{\psi})$, defined by (3), is magnitude hyperoctagonally symmetric if and only if the squared magnitudes of the numerator and denominator, $f(j\boldsymbol{\varphi})f(-j\boldsymbol{\varphi})$ and $g(j\boldsymbol{\varphi})g(-j\boldsymbol{\varphi})$, are individually hyperoctagonally symmetric.

The proof of Theorem 2 can be given in a similar way as for Theorem 1 in [13]. However, the proof itself is helpful for understanding the proofs of the following new theorems. We therefore sketch the proof of Theorem 2 here.

The sufficiency of the condition in Theorem 2 is obvious. For proof of necessity, consider that the statement of the theorem implies that

$$H(j\boldsymbol{\varphi})H(-j\boldsymbol{\varphi}) = H(jQ\boldsymbol{\varphi})H(-jQ\boldsymbol{\varphi}), \tag{5}$$

where Q is an $n \times n$ integer matrix which is obtained by reordering the row vectors of the unit matrix, causing a reordering of the components φ_i ($i = 1$ to n) of the vector $\boldsymbol{\varphi}$ in an arbitrary way. By analytic continuation, the statement (5) implies that

$$\frac{f(\boldsymbol{\psi})f(-\boldsymbol{\psi})}{g(\boldsymbol{\psi})g(-\boldsymbol{\psi})} = \frac{f(Q\boldsymbol{\psi})f(-Q\boldsymbol{\psi})}{g(Q\boldsymbol{\psi})g(-Q\boldsymbol{\psi})} \tag{6}$$

holds. On the left-hand side of (6), a cancellation cannot occur because $H(\boldsymbol{\psi})$ is assumed to be irreducible and free of allpasses. Therefore, it follows from (6) that

$$f(\boldsymbol{\psi})f(-\boldsymbol{\psi}) = Kf(Q\boldsymbol{\psi})f(-Q\boldsymbol{\psi}), \tag{7a}$$

$$g(\boldsymbol{\psi})g(-\boldsymbol{\psi}) = Kg(Q\boldsymbol{\psi})g(-Q\boldsymbol{\psi}) \tag{7b}$$

hold, where K could first in principle be a polynomial. This would imply that the numerator and denominator of the left-hand side of (6) has a common factor, which however has been excluded by the assumption. Hence, K is a constant. In fact, it is easy to see, by setting all the ψ_i, $i = 1$ to n, to be equal, that $K = 1$.

Note that, according to (2), the same statement also holds for the z-domain transfer function, and that the assumption of Theorem 2 can be extended by saying that (7) holds for an arbitrary matrix Q.

Applying Theorems 1 and 2, we state that the denominator $g(\boldsymbol{\psi})$ of an n-D MHOS reference domain transfer function must have the form

$$g(\boldsymbol{\psi}) = \gamma \prod_{i=1}^{n} g_0(\psi_i), \tag{8}$$

where $g_0(\psi_i)$ is a 1-D (strict) Hurwitz polynomial and γ is a unimodular constant.

B. Magnitude Hyperspherical Symmetry (MHSS) of $g(\boldsymbol{\psi})$

Theorem 3: An n-D allpass-free and irreducible reference domain transfer function $H(\boldsymbol{\psi})$, defined by (3), is magnitude hyperspherically symmetric if and only if the squared magnitudes of the numerator and denominator, $f(j\boldsymbol{\varphi})f(-j\boldsymbol{\varphi})$ and $g(j\boldsymbol{\varphi})g(-j\boldsymbol{\varphi})$, are individually hyperspherically symmetric.

The proof of this theorem follows the same strategy as above. In particular, the sufficiency is obvious. For proof of necessity, we consider that the statement of the theorem amounts to saying that the squared magnitude responses along two arbitrary radial lines in the n-D reference domain are identical. This is equivalent to stating that

$$H(j\boldsymbol{u}\varphi)H(-j\boldsymbol{u}\varphi) = H(j\boldsymbol{u}_1\varphi)H(-j\boldsymbol{u}_1\varphi) \tag{9}$$

holds, where $\boldsymbol{u}$ is an arbitrarily-chosen real unimodular vector, i.e., the Euclidian norm $\|\boldsymbol{u}\| = 1$, and where $\boldsymbol{u}_1 = (1, 0, \cdots, 0)^T$. That is, the behavior of the squared magnitude response on an arbitrary radial line is the same as that on the φ_1 axis. By analytic continuation, the statement (9) implies that

$$\frac{f(\boldsymbol{u}\psi)f(-\boldsymbol{u}\psi)}{g(\boldsymbol{u}\psi)g(-\boldsymbol{u}\psi)} = \frac{f(\boldsymbol{u}_1\psi)f(-\boldsymbol{u}_1\psi)}{g(\boldsymbol{u}_1\psi)g(-\boldsymbol{u}_1\psi)} \tag{10}$$

holds, where the argument $\boldsymbol{u}\psi$ can be simply considered as written in the hyperpolar coordinate system. Because the same cancellation discussion under Theorem 2 applies here, it follows from (10) that

$$f(\boldsymbol{u}\psi)f(-\boldsymbol{u}\psi) = Kf(\boldsymbol{u}_1\psi)f(-\boldsymbol{u}_1\psi), \tag{11a}$$

$$g(\boldsymbol{u}\psi)g(-\boldsymbol{u}\psi) = Kg(\boldsymbol{u}_1\psi)g(-\boldsymbol{u}_1\psi). \tag{11b}$$

As is shown above, the factor $K = 1$.

Note that the assumption of Theorem 3 can be extended by saying that (11) holds for an arbitrary vector $\boldsymbol{u}$. Although MHSS in the both $\boldsymbol{\varphi}$- and $\boldsymbol{\omega}$-domains are not exactly equivalent, Theorem 3 also holds for the z-domain transfer function. The proof can follow the same way as above, where the frequency variable $\boldsymbol{\omega}$ is transformed to $\boldsymbol{u}\omega$.

Because an n-D z-domain filter transfer function $H_d(z)$ is a trigonometric function in the frequency variables ω_i ($i = 1$ to n), or more precisely, a rational function in $\sin\omega_i$ and $\cos\omega_i$, the condition (11) can not hold exactly. However, a good approximation can usually be made in the $\boldsymbol{\varphi}$-domain. Therefore, we consider hereafter the reference domain transfer function $H(\boldsymbol{\psi})$, although both hyperspherical symmetries in the $\boldsymbol{\omega}$- and $\boldsymbol{\varphi}$-domain are not equivalent.

Theorem 4: The denominator polynomial $g(\boldsymbol{\psi})$ of an n-D allpass-free and irreducible reference domain transfer function $H(\boldsymbol{\psi})$, defined by (3), is magnitude hyperspherically symmetric if and only if it is given by (8)

$$g(\boldsymbol{\psi}) = \gamma \prod_{i=1}^{n} g_0(\psi_i),$$

where $g_0(\psi)$ (the frequency index is dropped for simplicity) is a 1-D (strict) Hurwitz polynomial whose degree tends to infinity, the squared magnitude $g_0(j\varphi)g_0(-j\varphi)$ converges to $e^{\alpha\varphi^2}$ within a multiplicative constant, and α is a constant.

Proof: The sufficiency of the assumption for the symmetric requirement is obvious. Furthermore, if $g_0(j\varphi)g_0(-j\varphi)$ converges to $e^{\alpha\varphi^2}$, which is greater than zero for all real φ, there exist no real roots of the real even polynomial $g_0(j\varphi)g_0(-j\varphi)$ in the variable φ. Substituting $j\varphi$ by ψ, we obtain the even polynomial $g_0(\psi)g_0(-\psi)$ in the variable ψ, that has then no roots on the imaginary axis of ψ. Further, a zero point ψ_0 of the polynomial $g_0(\psi)g_0(-\psi)$ always implies a zero point $-\psi_0$ of the same polynomial so that we can write this polynomial as a product of a Hurwitz and an anti-Hurwitz polynomial, say $g_0(\psi)$ and $g_0(-\psi)$.

For proof of necessity, because MHSS transfer functions are necessarily MHQS and MHOS, therefore, according to Theorems 1 and 2 and (8), we only need show that it is necessary that $g_0(j\varphi)g_0(-j\varphi)$ is of infinite order and converges to $e^{\alpha\varphi^2}$. For simplicity, we define the squared magnitude function

$$G_0(\varphi^2) = g_0(j\varphi)g_0(-j\varphi)$$

and say that

$$G_0(\varphi^2) = e^{\alpha\varphi^2}$$

must hold. According to Theorem 3, (11b) must hold, that means particularly for the 2-D case ($n = 2$) the following equations must hold for MHSS:

$$G_0(u_1^2\varphi^2)G_0(u_2^2\varphi^2) = G_0(\varphi^2)G_0(0), \tag{12a}$$

with

$$u_1^2 + u_1^2 = 1. \tag{12b}$$

We now establish the constraints that must therefore exist on the coefficients of the polynomial $G_0(\varphi^2)$. Let

$$G_0(\varphi^2) = \sum_{k=0}^{M} a_k \cdot (\varphi^2)^k, \tag{13}$$

where M is the degree of the polynomial $g(j\varphi)$. Then, we can express the right-hand side of (12a) as

$$G(\varphi^2)G(0) = \sum_{k=0}^{M} a_0 a_k \cdot (\varphi^2)^k \tag{14a}$$

and the left-hand side of (12a) as

$$\begin{aligned} &G_0(u_1^2\varphi^2)G_0(u_2^2\varphi^2) \\ &= \sum_{j=0}^{M}\sum_{i=0}^{M} a_i a_j u_1^{2i} u_2^{2j} (\varphi^2)^{i+j} \\ &= \sum_{j=0}^{M}\sum_{k=j}^{M+j} a_{k-j} a_j u_1^{2(k-j)} u_2^{2j} (\varphi^2)^k \\ &= \sum_{k=0}^{M}\sum_{j=0}^{k} a_{k-j} a_j u_1^{2(k-j)} u_2^{2j} (\varphi^2)^k \\ &\quad + \sum_{k=M+1}^{2M}\sum_{j=k-M}^{M} a_{k-j} a_j u_1^{2(k-j)} u_2^{2j} (\varphi^2)^k. \end{aligned} \tag{14b}$$

By comparing the coefficients of (14a) and the first term of (14b), we obtain the necessary condition for satisfying (12a)

$$a_0 a_k = \sum_{j=0}^{k} a_{k-j} a_j u_1^{2(k-j)} u_2^{2j}. \quad (15)$$

This equation is a recursive formula. In order to obtain an explicit formula, we consider the following cases:

1) Zero- and First-Order Case: It can be easily checked that, for $k = 0, 1$ and a_0, a_1 being arbitrary, (15) holds automatically.

2) Second-Order Case: For $k = 2$, (15) can be written as

$$a_0 a_2 = a_2 a_0 u_1^4 + a_1 a_1 u_1^2 u_2^2 + a_0 a_2 u_2^4,$$

from which we have, under consideration of (12b),

$$a_2 = \frac{a_1^2}{2a_0}$$

for arbitrary a_0 and a_1.

3) Arbitrary Order Case: For any nonnegative integer k, we now show that it is necessary that

$$a_k = \frac{a_1^k}{k! a_0^{k-1}}. \quad (16)$$

It is clear from above that for $k = 0, 1, 2$ (16) is necessary. Assume that (16) is necessary for a particular k, then we have a_{k+1} determined by

$$a_0 a_{k+1} = \sum_{j=0}^{k+1} a_{k+1-j} a_j u_1^{2(k+1-j)} u_2^{2j},$$

which leads to

$$a_0 a_{k+1}\left(1 - u_1^{2(k+1)} - u_2^{2(k+1)}\right) = \sum_{j=1}^{k} a_{k+1-j} a_j u_1^{2(k+1-j)} u_2^{2j}.$$

Using the binomial formula, we have

$$\begin{aligned}(u_1^2 + u_2^2)^{k+1} &= \sum_{j=0}^{k+1} \frac{(k+1)!}{(k+1-j)!j!} u_1^{2(k+1-j)} u_2^{2j} \\ &= \sum_{j=1}^{k} \frac{(k+1)!}{(k+1-j)!j!} u_1^{2(k+1-j)} u_2^{2j} \\ &\quad + u_1^{2(k+1)} + u_2^{2(k+1)}.\end{aligned}$$

Considering the relation (12b), we obtain a_{k+1} uniquely determined by

$$a_{k+1} = \frac{\sum_{j=1}^{k} a_{k+1-j} a_j u_1^{2(k+1-j)} u_2^{2j}}{a_0 \sum_{j=1}^{k} \frac{(k+1)!}{(k+1-j)!j!} u_1^{2(k+1-j)} u_2^{2j}}. \quad (17)$$

Putting the assumption (16) for a_{k+1-j} and a_j, respectively, into (17) yields

$$a_{k+1} = \frac{a_1^{k+1}}{(k+1)! a_0^k},$$

so it is proved that (16) is necessary for the case of arbitrary order k.

The second term of the right-hand side of (14b) is a term whose degree is higher than the degree M of the polynomial (14a). Thus, it is clearly the higher-order error term. This error term converges to zero as the degree M tends to infinity. Indeed, we have for $M \to \infty$ in (14b)

$$\begin{aligned}G_0(u_1^2\varphi^2) G_0(u_2^2\varphi^2) &= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} a_{k-j} a_j u_1^{2(k-j)} u_2^{2j} (\varphi^2)^k \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_{k-j} a_j u_1^{2(k-j)} u_2^{2j} (\varphi^2)^k,\end{aligned}$$

and the error term is zero.

By using (16), the function $G_0(\varphi^2)$ may be written in the form

$$G_0(\varphi^2) = \sum_{k=0}^{M} \frac{a_1^k}{k! a_0^{k-1}} (\varphi^2)^k,$$

and for the case $M \to \infty$

$$G_0(\varphi^2) = a_0 \sum_{k=0}^{\infty} \frac{\left(\frac{a_1}{a_0}\varphi^2\right)^k}{k!} = a_0 e^{\alpha\varphi^2} \quad \text{and} \quad \alpha = \frac{a_1}{a_0},$$

which reveals the necessary and sufficient condition on MHSS denominator polynomials. Q.E.D.

Theorem 4 implies that the squared magnitudes of the $\boldsymbol{\varphi}$-domain exactly MHSS denominator polynomials are separable in their individual frequency variables and each 1-D factor is of infinite order, converging to the exponential function $a_0 e^{\alpha\varphi^2}$. That is, except for a scaling constant, the ideal denominator polynomial is fully determined by only one parameter, namely, α.

III. Necessary and Sufficient Conditions on the Numerator $f(\boldsymbol{\psi})$ of MD Magnitude Hyperspherically Symmetric Transfer Functions

The numerator polynomials $f(\boldsymbol{\psi})$ of the MD MHSS transfer functions do not have to be separable in their frequency variables. Therefore, we expect somewhat less restrictive conditions. However, according to Theorem 3, the numerator $f(\boldsymbol{\psi})$ must be MHSS.

Theorem 5: An n-D reference domain polynomial $f(\boldsymbol{\psi})$ is magnitude hyperspherically symmetric if and only if it is *polynomial* hyperspherically symmetric.

Proof: The sufficiency is obvious. For proof of necessity, consider that the statement of the theorem amounts to saying that, by analytic continuation,

$$f(\boldsymbol{u}\psi) f(-\boldsymbol{u}\psi) = f(\boldsymbol{u}_1\psi) f(-\boldsymbol{u}_1\psi) \quad (18)$$

holds, where $\boldsymbol{u}$ and $\boldsymbol{u}_1$ are defined as in Theorem 3. Equation (18) says that each factor of $f(\boldsymbol{u}\psi)$ must be a factor either of $f(\boldsymbol{u}_1\psi)$ or of $f(-\boldsymbol{u}_1\psi)$. Consequently, every factor of $f(\boldsymbol{u}\psi)$ and thus $f(\boldsymbol{u}\psi)$ itself has to be independent of any particular choices of $\boldsymbol{u}$, which leads to

$$f(\boldsymbol{u}\psi) = f(\boldsymbol{u}_1\psi)$$

or equivalently that $f(\boldsymbol{u}\psi)$ is *polynomial hyperspherically symmetric* (PHSS). Q.E.D.

Note that we can also conclude $f(\boldsymbol{u}\psi) = f(-\boldsymbol{u}_1\psi)$ from (18), which means that an n-D reference domain PHSS polynomial is an even function in each reference frequency variable and fully determined by its behavior on one frequency axis. Moreover, the general form of reference domain MHSS polynomials is a reference domain PHSS polynomial given as $f\ (\Sigma_{i=1}^{n}\psi_i^2)$. That is, all the polynomials of this kind can be obtained by using frequency transformation techniques.

Theorems 4 and 5 impose strict conditions on the form of MHSS filter transfer functions. However, good symmetric filters can nevertheless be designed, as shown in the next section and in [16], [17].

IV. An Empirical Algorithmic Design Procedure for MD Magnitude Hyperspherically Symmetrical Low-Pass Filters

According to Theorems 4 and 5, the general n-D squared magnitude response of MHSS filter transfer functions is uniquely determined by its behavior on one frequency axis, which can be given as

$$M_0^2(\varphi) = H_0(j\varphi)H_0(-j\varphi) = \frac{F_0(\varphi^2)}{G_0(\varphi^2)}.$$

In order to obtain a nearly-optimal design, the denominator $G_0(\varphi^2)$ *must* be chosen to approximate the exponential function $e^{\alpha\varphi^2}$. As a very simple possibility, we set it to be a finite Taylor series of the exponential function [16]:

$$G_0(\varphi^2) = E_{\alpha,M}(\varphi^2) := \sum_{k=0}^{M} \frac{(\alpha\varphi^2)^k}{k!},$$

where M is the degree of the corresponding denominator polynomial of the transfer function in one frequency direction and α is a free parameter. The numerator $F_0(\varphi^2)$ must be a squared form of an even polynomial in φ, that can be determined in the same way as shown in [16], [17]. However, combining the methods of [16], [17] can give a design approach which, on the one hand, involves very few free parameters so that heuristic design methods can be used successfully and, on the other hand, guarantees a short transition band. In this sense, we choose the numerator

$$F_0(\varphi^2) = E_{\alpha/2,L}^2(\varphi^2)\prod_{i=1}^{N}(1 - \varphi^2/\varphi_{0i}^2)^2,$$

where L is an integer, denoting the degree of the Taylor series approximating the exponential function $e^{(\alpha/2)\varphi^2}$ in the numerator, and φ_{0i} ($i = 1$ to N) determine N positive zero points of the magnitude response. For low-pass filters, L should be in the interval $0 \le L \le \frac{1}{2}M - N$ in order to ensure sufficient stop-band loss.

If φ_{0i} is appropriately located, the (squared) magnitude response $M_0^2(\varphi)$ approximates 1 in the low frequency region or pass band, because the zero-point factors approach 1 and the squared Taylor series $E_{\alpha/2,L}^2(\varphi^2)$ in the numerator approximates the same exponential function $e^{\alpha\varphi^2}$ as closely as does the Taylor series $E_{\alpha,M}(\varphi^2)$ in the denominator. For higher frequencies, $E_{\alpha,M}$ becomes increasingly greater than $E_{\alpha/2,L}^2$, the magnitude response begins to enter the stop band. In order to force the (squared) magnitude response $M_0^2(\varphi)$ to reach the stop band faster, the numerator is assigned to have zero points created by the factors $(1 - \varphi^2/\varphi_{0i}^2)^2$. In this way, a desirable stop-band loss can be achieved without choosing a large degree difference of M and L, which amounts to a reduction of filter degree. The parameters to be determined for a given filter specification are M, L, α, and φ_{0i}. Hereafter, we give a heuristic procedure for determining these parameters. For simplicity, we discuss only the case where there is only one zero point, because the case of one zero point may satisfy most practical requirements and the more general case can be considered in a similar way. Indeed, an additional zero point can always be added at the relative maximum point in the stop band.

A. 1-D Design Procedure

Let a low-pass filter specification be given as: The minimum value of the squared magnitude response $M_0^2(\varphi)$ in the pass-band is A_1, the pass-band edge is Ω_1; the maximum value of $M_0^2(\varphi)$ in the stop-band is A_2, the stop-band edge is Ω_2. The design procedure can be divided into the following steps:

Step 1: Estimate the degree M and set the degree L to be the greatest integer smaller than $\frac{1}{2}M - 1$, then go to Step 2.

According to the special choice of the numerator $F_0(\varphi^2)$, the degree M should be chosen with respect to the prescribed pass-band radian Ω_1. The necessary degree M for a particular pass-band radian can, for example, be examined by statistical methods. The parameter L can certainly affect the pass-band radian, too. However, we set it to depend on M in order to get a lower overall filter degree. Certainly, the heuristically chosen degree M can be optimized, step by step. Usually, the value of M is smaller than 30.

Step 2: Estimate α, then go to Step 3.

The value of α is essentially determined by the width of the transition band $\Omega_2 - \Omega_1$ and the prescribed maximum response A_2 in the stop band. Initial values between 20 and 30 can always offer quite satisfactory results.

Step 3: Estimate φ_0, then go to Step 4.

The zero point is properly set to be located in the near from the stop-band edge so that the stop-band maximum response A_2 cannot be exceeded.

Step 4: Determine the increase of α for the numerator, then, if necessary, go to a step above.

The α value in the numerator factor $E^2_{\alpha/2,L}(\varphi^2)$ should be finally a little greater than that in the denominator $E_{\alpha,M}(\varphi^2)$ in order to take into account the influence of the zero-point factor in the numerator within the pass band, guaranteeing the minimum response A_1 in the whole pass band. We indicate this difference in the Taylor series of the numerator and denominator by using α' for the numerator factor $E_{\alpha'/2,L}$, instead of α.

B. MD Frequency Transformation

The above design procedure has been effectively implemented, using the Mathematica 2.0 software package. Our experience shows that the squared magnitude response $M_0^2(\varphi)$ is not very sensitive with respect to the parameters M, L, α, and φ_0 so that this empirical procedure gives a fast design method leading to desired results.

After the parameters M, L, α, and φ_0 are obtained, a frequency transformation

$$\varphi^2 = \tau(\boldsymbol{\varphi}^2), \qquad (19)$$

where $\boldsymbol{\varphi}^2 = (\varphi_1^2, \varphi_2^2, \cdots, \varphi_n^2)^T$, is applied to the numerator $F_0(\varphi^2)$. The resultant transfer function is then given by

$$H(\boldsymbol{\psi}) = \frac{E_{\alpha'/2,L}(\tau(-\boldsymbol{\psi}^2))\prod_{i=1}^{N}\left(1 - \tau(-\boldsymbol{\psi}^2)/\varphi_{0i}^2\right)}{\prod_{j=1}^{n} g_0(\psi_j)},$$

where $\boldsymbol{\psi}^2 = (\psi_1^2, \psi_2^2, \cdots, \psi_n^2)^T$ and $g_0(\psi_j)$ is a 1-D (strict) Hurwitz polynomial obtained in the way described in Theorem 4.

To achieve MHSS in the reference frequency domain $\boldsymbol{\varphi}$, which approximates the same symmetry in the frequency domain $\boldsymbol{\omega}$, the frequency transformation (19) must be chosen as

$$\varphi^2 = \sum_{i=1}^{n} \varphi_i^2. \qquad (20a)$$

For better symmetry in the frequency domain $\boldsymbol{\omega}$, other frequency transformations have been proposed. An often used transformation is the so-called McClellan transformation [9], [10]

$$(1 + \varphi^2) = \prod_{i=1}^{n} (1 + \varphi_i^2), \qquad (20b)$$

which gives a better MHSS in the $\boldsymbol{\omega}$-domain. Also, a generalized McClellan transformation has been proposed [10]

$$(1 + \beta\varphi^2) = \prod_{i=1}^{n} (1 + \beta\varphi_i^2), \qquad (20c)$$

where β is a real parameter. Using an appropriate β, improved $\boldsymbol{\omega}$-domain MHSS in the region somewhat far from the origin can be achieved.

The factor $E_{\alpha'/2,L}(\varphi^2)$ of the numerator of the transfer function before frequency transformation can be factorized in first- and second-order factors in φ^2. In order to obtain better MHSS in certain region of the $\boldsymbol{\omega}$-domain, the generalized McClellan transformation (20c) can be applied to each factor of $E_{\alpha'/2,L}(\varphi^2)$, where different values β may be chosen for distinct factors [19], and the optimum value of each β can be obtained by using an optimization procedure.

C. Reduction of Filter Degree

The approximation method chosen for the empirical design is a maximal flat approach that provides a satisfactory result only for an appropriate high-degree of the transfer function. In order to reduce the degree of the filter, the general form of the numerator $F_0(\varphi^2)$ and the denominator $G_0(\varphi^2)$ as well should be employed, where 1-D optimization procedures are used for determining the coefficients of the general form numerator and denominator polynomials. The approximation approach can follow the following steps:

Step 1: Let the polynomials $E_{\alpha'/2,L}(\varphi^2)$ in the numerator be replaced by an even polynomial of the general form and determine its coefficients and the parameters φ_{0i} and α of the other terms in the numerator and denominator by using a 1-D optimization program, where the coefficient values given by the empirical design are used as the initial values of optimization. The optimization reference is the given radial filter specifications.

Step 2: Let the denominator $E_{\alpha,M}(\varphi^2)$ be replaced by an even polynomial of the general form and determine the coefficients of the new denominator $G_0(\varphi^2)$ by using a 1-D optimization program, where the α value given by Step 1 is used to build the initial coefficient values of optimization. The optimization reference is the exponential function $e^{(\alpha/4)\omega^2}$ in the ω-domain, where the α is given by Step 1.

Step 3: The general form numerator and denominator from Steps 1 and 2 build the final radial squared magnitude response $M_0^2(\varphi)$. The numerator $F_0(\varphi^2)$ of this $M_0^2(\varphi)$ is further optimized by using a 1-D optimization program, where the corresponding coefficient values from Step 1 are used as initial values of optimization. The optimization reference is the given radial filter specifications.

Step 1 of the above approach serves to find a better set of initial coefficients of the numerator for the optimization in Step 3 as well as to find the optimal value for α of the maximal flat form denominator, whereas Step 2 determines the optimal coefficients of the general form denominator, that is, Step 2 is a denominator only optimization. In particular, the MHSS symmetry of the denominator in the critical frequency region can be guaranteed by the Step 2 optimization. In Step 3, the numerator coefficients are adjusted to the general form denominator. The improvement achieved by this optimization approach is illustrated by examples in the next section.

V. Design Examples

For the first two of the following low-pass filter examples, the above empirical design procedure is applied, where the constants T_i in the definition of the variables z_i ($i = 1$ to n) (cf. (1)) are normalized to be 1. For simplicity the frequency transformation (20a) is utilized for Examples 1 and 3. In Example 2, the transformation (20a) is also applied to the term $E_{\alpha'/2,L}$, whereas a more general transformation based on (20c) is used for the zero factor term $(1 - \varphi^2/\varphi_0^2)$, in order to improve the symmetry in the transition band. The squared magnitude responses have then the form

$$M_d^2(\boldsymbol{\omega}) = M^2(\boldsymbol{\varphi})|_{\varphi_i = \tan(\omega_i/2),\, i=1 \text{ to } n}$$

$$= \frac{E^2_{\alpha'/2,L}\left(\sum_{i=1}^{3} \varphi_i^2\right)\left(1 - \left(\sum_{i=1}^{3} \varphi_i^2 + \beta(\varphi_1^2\varphi_2^2 + \varphi_2^2\varphi_3^2 + \varphi_3^2\varphi_1^2)\right)/\varphi_0^2\right)^2}{\prod_{i=1}^{3} E_{\alpha,M}(\varphi_i^2)}.$$

In all the examples, very good magnitude hyperspherical symmetry has been achieved with moderate filter degree.

Example 1: A 3-D *low degree*, say fifth degree in each frequency variable, magnitude spherically symmetric low-pass filter is designed to meet the specification: $A_1 = 0.99$ at the edge frequency $\Omega_1 = 0.2$, and $A_2 = 0.001$ at the edge frequency $\Omega_2 = 0.875$. The obtained parameters are $M = 5$, $L = 1$, $\alpha = 35.5$, $\alpha' = 47.5$, and $\Omega_0 = 0.92$ ($\varphi_0 = \tan(\Omega_0/2)$). In the frequency transformation of the numerator, the term with the factor β is omitted according to what is said above. The contour diagram and perspective view for the cross section $\omega_3 = 0$ and a 3-D surface plot of the corresponding squared magnitude response $M_d^2(\boldsymbol{\omega})$ is shown in Fig. 1.

Example 2: A 3-D *wide-band* magnitude spherically symmetric low-pass filter is designed to meet the specification: $A_1 = 0.995$ at the edge frequency $\Omega_1 = 0.85$, and $A_2 = 0.001$ at the edge frequency $\Omega_2 = 1.635$. The obtained parameters are $M = 20$, $L = 8$, $\alpha = 28.5$, $\alpha' = 30.2$, and $\Omega_0 = 1.33$ ($\varphi_0 = \tan(\Omega_0/2)$). In the frequency transformation of the numerator, the factor β is chosen to 1.33. The contour diagram and perspective view for the cross section $\omega_3 = 0$ and a 3-D surface plot of the corresponding squared magnitude response $M_d^2(\boldsymbol{\omega})$ is shown in Fig. 2.

Example 3: A 2-D magnitude circularly symmetric low-pass filter having the same radial magnitude specifications as in Example 2 is designed to demonstrate the degree reduction technique, where only the first step of the optimization approach has been done, because for this particular example this step brings out the most improvement. The radial squared magnitude response $M_0^2(\varphi)$ is chosen as

$$M_0^2(\varphi) = \frac{F_0(\varphi^2)}{G_0(\varphi^2)} = \frac{\left(1 + \sum_{i=1}^{L} b_i \varphi^{2i}\right)^2 (1 - \varphi^2/\varphi_0^2)^2}{E_{\alpha,M}(\varphi^2)}.$$

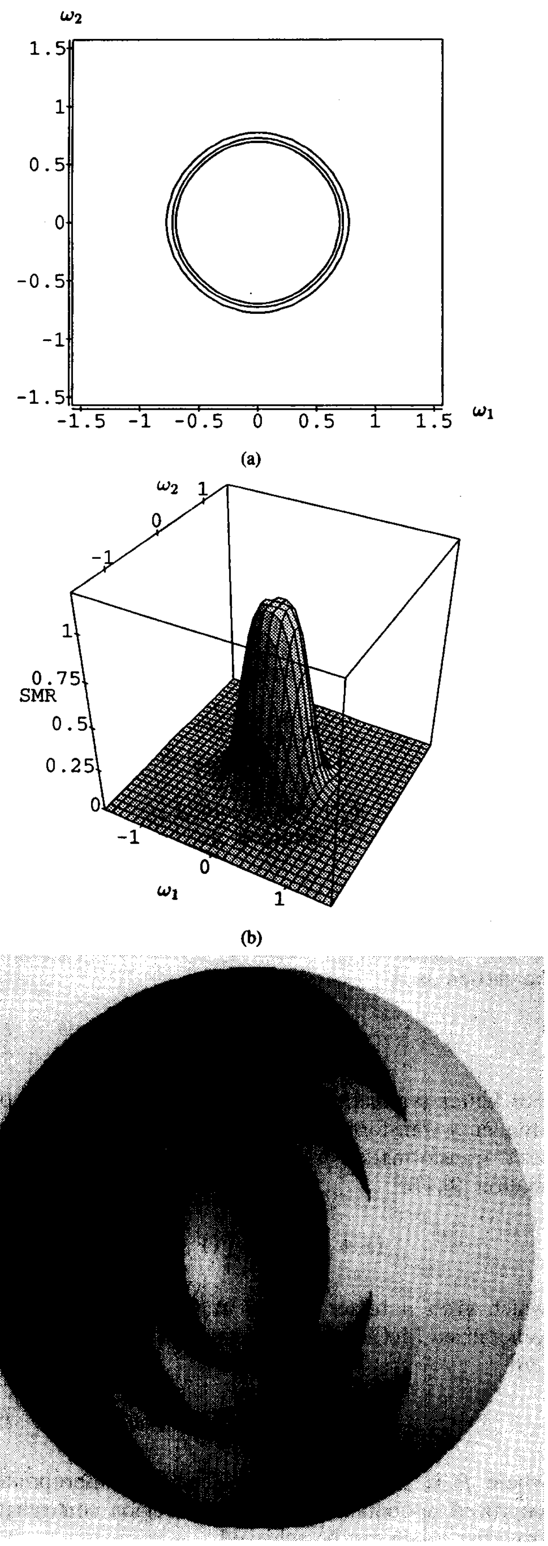

Fig. 1. Squared magnitude response (SMR) in Example 1. (a) Contour diagram (cross section $\omega_3 = 0$). (b) Perspective view (cross section $\omega_3 = 0$). (c) Surface plot (SMR = 0.9; 0.5; 0.1).

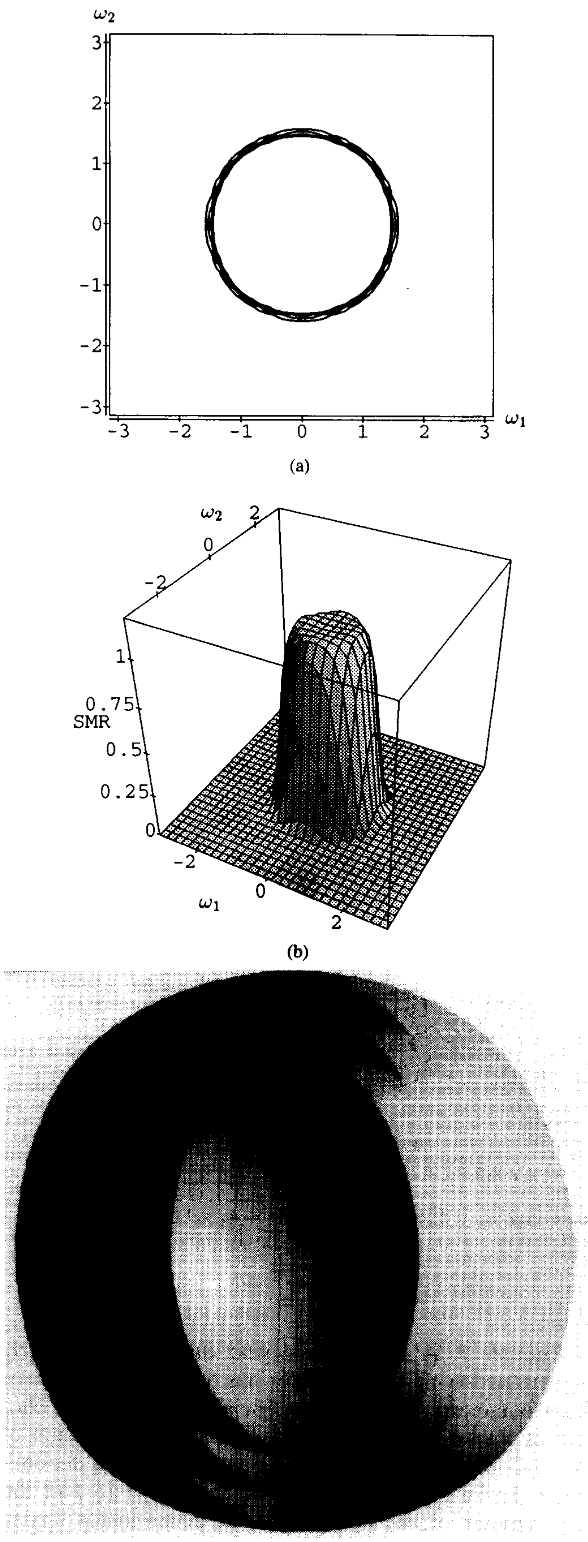

Fig. 2. Squared magnitude response (SMR) in Example 2. (a) Contour diagram (cross section $\omega_3 = 0$). (b) Perspective view (cross section $\omega_3 = 0$). (c) Surface plot (SMR = 0.9; 0.5; 0.1).

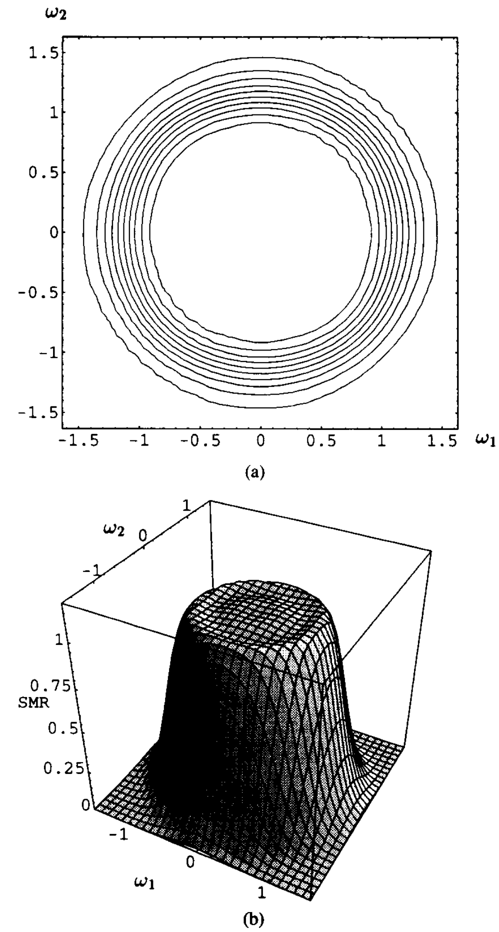

Fig. 3. Squared magnitude response (SMR) in Example 3. (a) Contour diagram. (b) Perspective view.

The obtained filter coefficients are listed in Table I. The perspective view and contour diagram are shown in Fig. 3. Although the filter degree M is only 9, the same filter specification of Example 2 has been achieved.

Example 4: A 2-D *narrow-band* magnitude circularly symmetric digital filter is designed for generating fractal images [20]. For such a filter, the radial magnitude response should run off linearly against the logarithm of frequency. Correspondingly, the numerator of the squared magnitude response is chosen to be a general even function as the product of first- and second-order factors

$$F_0(\varphi^2) = \prod_{i=1}^{L} (1 + b_i \varphi^2 + c_i \varphi^4)^2,$$

while the denominator is still chosen as the Taylor series $E_{\alpha, M}(\varphi^2)$. The parameters of this design example have been obtained by using an optimization procedure and are listed in Table II. The perspective view and contour diagram as well as the radial squared magnitude response $M_d^2(\boldsymbol{\omega})$ in the logarithmic measure are shown in Fig. 4.

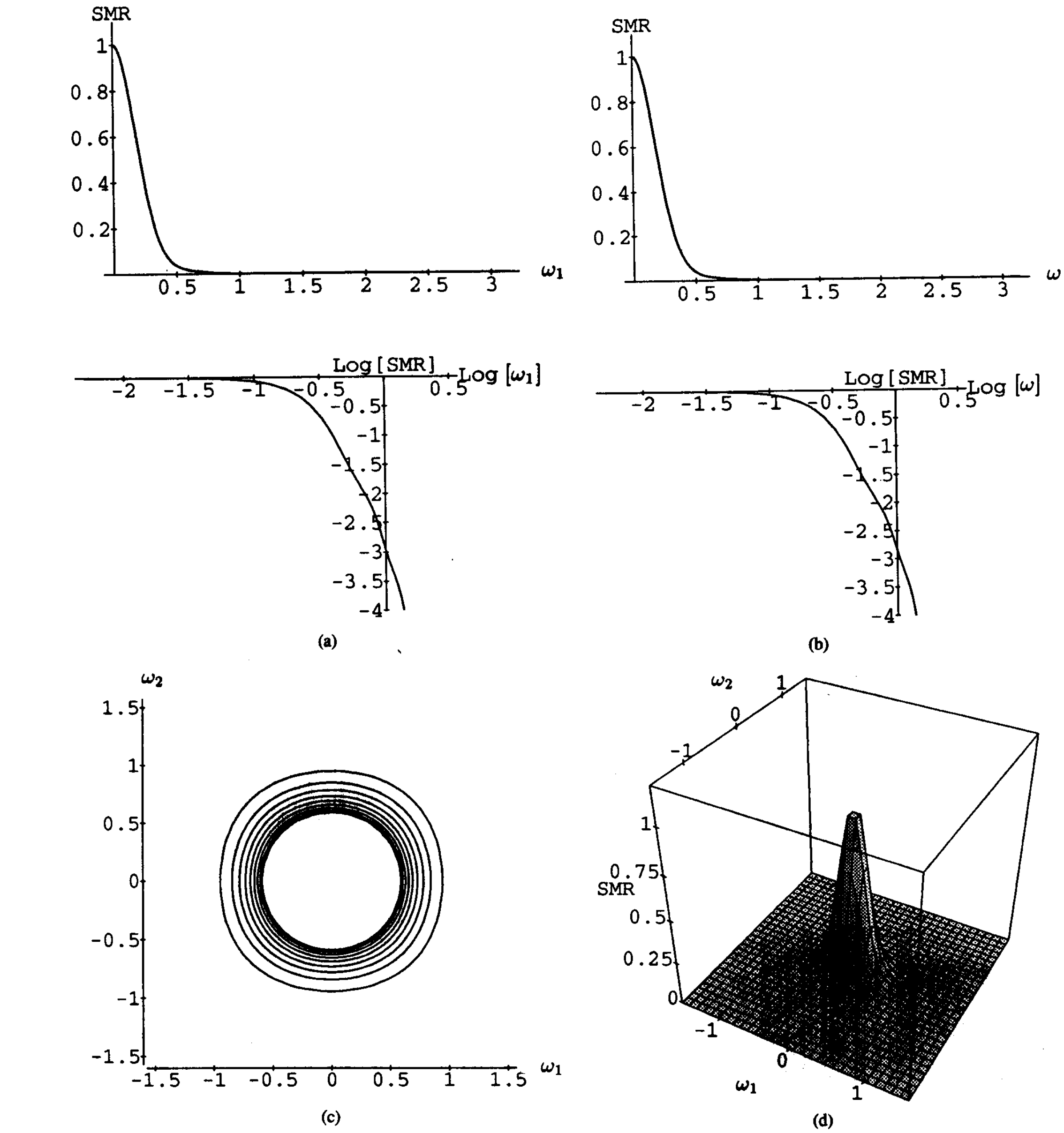

Fig. 4. Squared magnitude response (SMR) in Example 4. (a) Cross section $\omega_2 = 0$. (b) Cross section $\omega_1 = \omega_2$. (c) Contour diagram. (d) Perspective view.

TABLE I
FILTER PARAMETERS OF EXAMPLE 3

$M = 9$	$L = 3$	$\alpha = 17.3707$
$b_1 = 9.94327$	$b_2 = 21.3255$	$b_3 = 329.184$
$\varphi_0 = 1.25748$		

TABLE II
FILTER PARAMETERS OF EXAMPLE 4

$M = 16$	$N = 3$	$\alpha = 28$
$b_1 = -5.5$	$b_2 = -2.3$	$b_3 = -0.01$
$c_1 = 22.6875$	$c_2 = 1.9176$	$c_3 = 5.5 \cdot 10^{-5}$

Example 5: A 2-D narrow-band magnitude circularly symmetric low-pass digital filter has been designed, where both numerator and denominator are *separable* and chosen as Taylor series. Thus, this is a cascaded version of two 1-D filters. The order of the numerator and denominator Taylor series are $M_1 = 10$ and $M_2 = 16$ and the α-parameter of the numerator and denominator Taylor series are $\alpha_1 = -20$ and $\alpha_2 = 28$. The perspective view and contour diagram as well as the radial squared magnitude response in the logarithmic measure are drawn in Fig. 5, which show that even such a separable solution

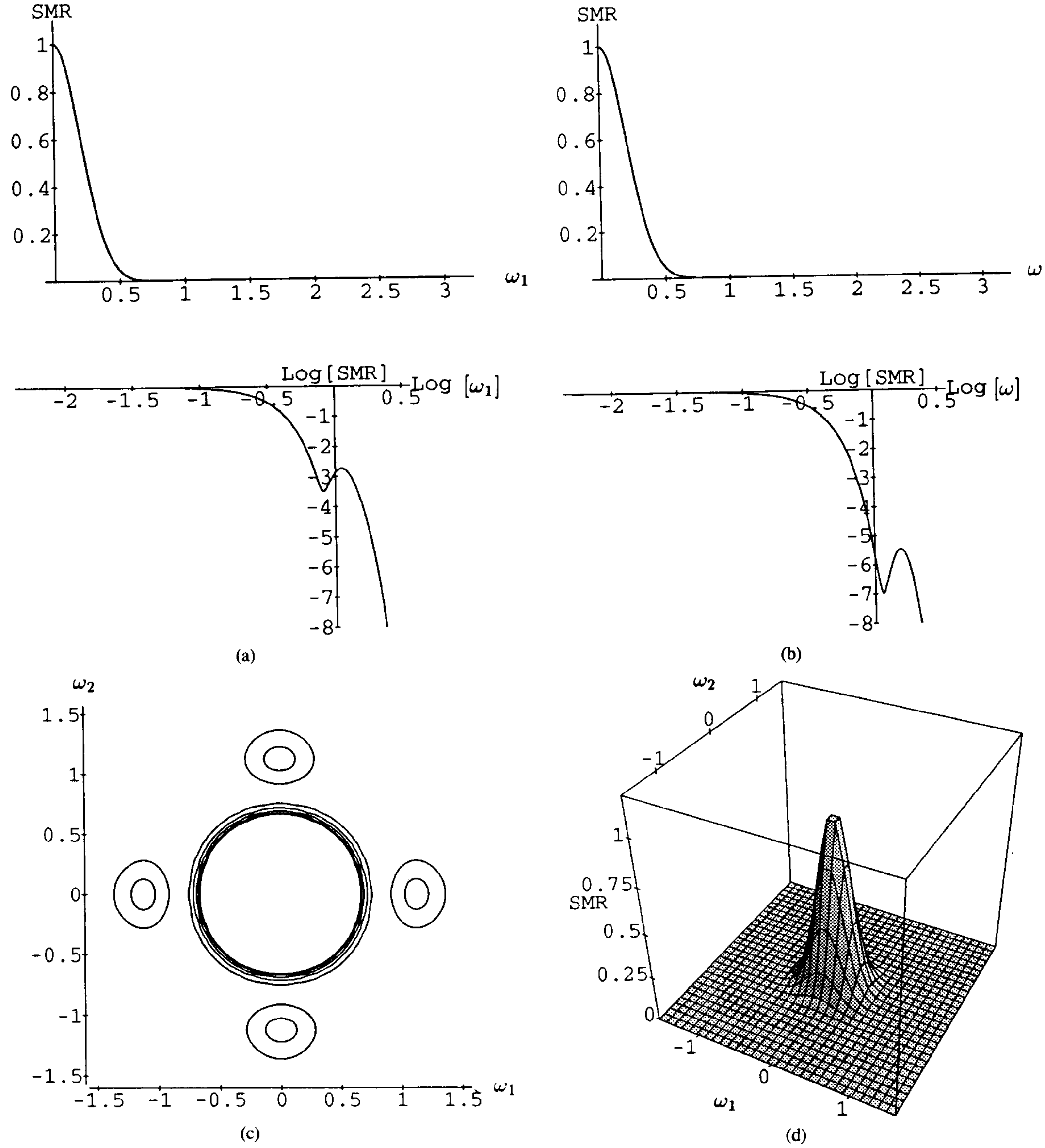

Fig. 5. Squared magnitude response (SMR) in Example 5. (a) Cross section $\omega_2 = 0$. (b) Cross section $\omega_1 = \omega_2$. (c) Contour diagram. (d) Perspective view.

possesses good circular symmetry in the region near the origin.

VI. Conclusions and Remarks

The major concern of this paper is the necessary and sufficient condition for magnitude hyperspherically symmetrical (MHSS) filter transfer functions. This condition is derived for both the numerator and denominator of the reference domain filter transfer function. It leads to the reference domain result that the squared magnitude of the denominator polynomial must be a product of 1-D polynomials approximating the squared exponential function and the numerator polynomial itself must be a 1-D radial even function. Therefore, the approximation procedure involved for the design of MHSS filters is essentially a 1-D procedure, and thus relatively easy to implement. The constraints imposed on the filter design are quite strict, but low-degree filters can nevertheless be designed.

It has been shown that there exist no exact MHSS filter transfer functions in the $\boldsymbol{\omega}$-domain. Approximations of such symmetry in the $\boldsymbol{\omega}$-domain can be achieved by using different $\boldsymbol{\varphi}$-domain frequency transformations. Also distinct frequency transformations can be utilized for a single transfer function. The parameters obtained by using the proposed 1-D design approach for MHSS filters can be further used as a set of very good initial values for subsequent additional MD numerical optimization procedures. However, such an MD procedure is usually not necessary because of the good results of the 1-D design approach.

It has been shown by Example 5 in Section V that for narrow-band filters, transfer functions having both separable numerator and separable denominator can be designed to possess certain MHSS symmetry in the near from the origin. Such a separable design may be sufficient for certain applications. If stricter symmetry is needed in some region far from the origin, the general form numerator and denominator should be employed (cf. Example 4).

It is observed that, if the numerator $E^2_{\alpha'/2,L}(\varphi^2)\prod_{i=1}^N(1 - \varphi^2/\varphi_{0i}^2)^2$ is replaced by $E_{\alpha'',L}(\varphi^2)\prod_{i=1}^N(1 - \varphi^2/\varphi_{0i}^2)$, the same squared magnitude response $M_0^2(\varphi)$ can be achieved. The degree required in this way would be much lower than before. However, one cannot find an MD polynomial whose squared magnitude response meets the frequency transformed MHSS version of this low-degree numerator $E_{\alpha'',L}(\tau(\varphi^2))\prod_{i=1}^N(1 - \tau(\varphi^2)/\varphi_{0i}^2)$. Indeed, Theorem 5 excludes such possibilities.

The reduction of the filter degree can be achieved by using the general form numerator $F_0(\varphi^2)$ as in Examples 3 and 4, instead of the maximal flat approximation for the numerator. The examples have shown that the improvement is significant. Further, the degree of the MHSS symmetry of the magnitude response can be improved if a general form denominator that approximates the ω-domain squared exponential function is employed. It is also worth mentioning that all these improvement methods need only 1-D approximations.

Comparing our results with prior results in the literature, we find that most previously designed MHSS filters are *nonrecursive* and are easier to design. Among the remaining *recursive* filters, most of them are semicausal and require reversal of the recursion directions for implementation. Some published good design examples of causal recursive MHSS filters exist but do not provide the filter coefficients (cf. [4]). Comparison of our results with the available causal recursive filters shows that the proposed method guarantees a better MHSS symmetry even for high-selective wide-band filters, that is very difficult for other direct optimization methods (cf. [1]–[3], [5]). The achievable filter degree in this paper is much lower than that of the related methods (cf. [16], [17]).

For high performance implementations of the obtained MHSS transfer functions, several decomposition techniques can be utilized. In particular, the filter can be implemented by using only 1-D second-order sections [17]. Also, the phase equalization problem of this filter is easy to solve [17], because the numerator of the transfer function is an even polynomial and the denominator is a product of 1-D polynomials.

VII. Acknowledgments

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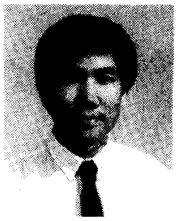

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