Simulation of Fractal Multidimensional Images Using Multidimensional Recursive Filters

L. T. Bruton and N. R. Bartley

Abstract—Fractal multidimensional (MD) images may be generated by simulating MD fractal Brownian motion (fBm). This is usually achieved by appropriately weighting the magnitudes of the Fourier coefficients of the MD discrete Fourier transform (DFT) of the required fractal image. In this contribution, it is proposed that MD hyperspherically-symmetric recursive filters be used to approximate fractal images directly in the spatial domain, thereby allowing spatially-variant characteristics to be obtained without the undesirable edge effects of the MD DFT method. The method is used to generate realistic 2D fractal landscape images having spatially-variant characteristics.

I. INTRODUCTION

Mandelbrot's FRACTAL GEOMETRY [1] has found a wide variety of applications. The statistical self-similarity of fractal shapes is inherent in the natural world and may be exploited to simulate natural processes. In [2], Voss describes the self-similarity of fractal shapes and distinguishes between determinstic fractals, such as the von Koch snowflakes, and random fractals. In particular, random fractional Brownian motion (fBm) corresponds to fractal landscapes and occurs widely in nature, accurately describing such phenomena as pitch variation in music, flesk noise in solid-state devices, 2D mountain landscapes and 3D spatio-temporal image sequences of moving formations of clouds [2].

The analysis and synthesis of 1D fBm has been of growing interest recently, and recent publications have appeared [3], [4] that explore fBm using wavelet-based techniques. In this contribution, it is shown that multidimensional (MD) fBm may be simulated in the spatial domain by applying MD noise, having a known MD spectral density function, to a MD hyperspherically-symmetric recursive digital filter.

Following [2], a MD continuous-domain signal \( x(t), t \in R^M \), possesses fBm if, throughout its domain, the MD difference \( \Delta(t_1, t_2) = x(t_1) - x(t_2) \) satisfies the following stochastic properties; for any \( t_1, t_2, t_1 \neq t_2 \), in the domain of \( x(t) \),

1) \( \Delta(t_1, t_2) \) is a zero-mean function having a Gaussian amplitude probability density function and,

2) the variance \( \sigma(t_1, t_2) \) of \( \Delta(t_1, t_2) \) is given, for some constant \( K \), by

\[
\sigma(t_1, t_2) = K||\Delta(t_1, t_2)||^H, H, K \in R
\]

where \( ||\Delta(t_1, t_2)|| \) is the MD Euclidean norm. The scaling parameter \( H \) controls the relative “roughness” (or “smoothness”) of the continuous-domain MD fractal surface \( x(t) \).

The concept of fractal dimension \( D \) is often used [2] to characterize fractal surfaces, where

\[
D = M + 1 - H
\]

and where \( M \) is the dimensionality of \( x(t) \). Generally, the fractal dimension \( D \) is in the range \( M \leq D \leq M + 1 \), corresponding to \( 0 \leq H \leq 1.0 \). For example, in the 2D case \((M = 2)\), it has been found that \( H \approx 0.8 \) corresponds to excellent 2D surface simulations of mountainous landscapes. The fractal dimension of the landscape is then given by

\[
D = 2 + 1 - 0.8 = 2.2.
\]

Let \( X{j\omega}, \omega \in R^M \), be the MD Fourier transform of \( x(t) \) so that \( \Phi(\omega) \equiv X{j\omega}X(-j\omega) = |X(j\omega)|^2 \) is the real continuous-domain MD spectral density function of \( x(t) \). It is well known that [2], [6] this MD fBm has the spectral density function

\[
\Phi(\omega) = \Phi_0 ||\omega||^{2H+1}, \Phi_0 \text{ constant } \in R
\]

SPECTRAL DENSITY FUNCTION OF MD fBm

where \( ||\omega|| \) is the Euclidean norm of the MD vector \( \omega \); that is, \( ||\omega|| \) is the Pythagorean distance of \( \omega \) from the origin in \( R^M \). The parameter \( \beta \), where

\[
\beta \equiv 2H + 1
\]

is referred to as the spectral exponent of the fBm, so that

\[
\Phi(\omega) = \Phi_0 ||\omega||^{-\beta}, \Phi_0, \beta \in R.
\]

Typically, for the case of a 2D landscape, one might choose \( H = 0.8 \), implying \( \Phi(\omega) = \Phi_0 ||\omega||^{-2.6} \) and \( |X(\omega)| = \Phi_0^{1/2} ||\omega||^{-1.3} \).

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The authors are with the Department of Electrical and Computer Engineering, University of Calgary, Calgary, Alberta, Canada.

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II. REVIEW OF THE SIMULATION OF MD fbM USING DIRECT SPECTRAL SYNTHESIS AND THE DISCRETE FOURIER TRANSFORM

We are concerned with a discrete-domain sampled version of \( x(t) \), which we write as \( x(n) \), \( n \in Z^M \), where the values of \( x(n) \) are quantized in machine-representable form. Clearly, \( x(n) \) can only be a discrete-domain quantized-amplitude approximation of a continuous-domain signal \( x(t) \); this follows because the statistical distributions of \( \Delta(n_1, n_2) \equiv x(n_1) - x(n_2) \) are not defined for non-integer values of \( n_1 \) and \( n_2 \) and one cannot consider a corresponding image surface to have any meaning at spatial resolutions near or below one sample distance. However, at much larger MD spatial distances where \( ||n_1 - n_2||_2 \gg 1 \), the continuous surface (obtained by connecting rectangularly adjacent samples of \( x(n) \) with planar surfaces) turns out to provide an excellent approximation to the continuous-domain fbM function \( x(t) \). This is achieved by ensuring that the discrete-domain spectral density function \( \Phi(\Omega) \), \( \Omega \in Z^M \), approximates (3) over an appropriate region of support. The DFT has been widely used for this purpose [2], [6]. We write the DFT of \( x(n) \) as \( X(\Omega) \) and assume MD regions of support for \( x(n) \) and \( X(\Omega) \) given by

\[
Z^M \equiv \{ n | M_i/2 \leq n_i \leq (M_i - 1)/2, \forall n_i, i = 1..M \}
\]

A. Direct Spectral Synthesis

Discrete-domain MD fbM signals are obtained by choosing the discrete-domain MD complex coefficients \( X(\Omega) \), such that

\[
E[|X(\Omega)|] = \frac{B}{||\Omega||^3}, \forall \Omega \in Z^M
\]

where \( E \) denotes expected value, and such that \( \arg X(\Omega) \) is randomly chosen with uniform probability in the interval \([0, 2\pi]\) [6]. The inherent conjugate symmetries of \( X(\Omega) \) in \( Z^M \) require that \( |X(\Omega)| \) be computed only \( 1/2^M \) of the discrete points in \( Z^M \). At the \( M \)-tuple \( \Omega \equiv \{ \Omega_1, \Omega_2, \ldots, \Omega_M \} \), the magnitude of the corresponding coefficient \( |X(\Omega)| \) is given by [6]

\[
|X(\Omega)| = \left| \frac{\text{Gauss}}{||\Omega||^3} \right|
\]

where Gauss is a procedure call that generates real numbers (each time the procedure is called) having zero mean and unity variance. For the 2D case, the corresponding inverse 2D DFT, \( x(n_1, n_2) = \text{DFT}^{-1}[X(\Omega)] \), has been shown to approximate excellent natural-looking surface landscapes and is the basis for the computer-based simulation of fractal-based scenes.

The above direct spectral synthesis method does have certain practical limitations. Obviously, to achieve the required spectral distribution, it is necessary to choose a sufficiently large region of support \( Z^M \); typically, the image size \( M_i \) in the \( i \)th dimension must exceed \( 512 \) sample points, for all dimensions \( i = 1, 2, \ldots, M \). In the 2D case, this corresponds to \( 512^2 = 262,144 \) data points per block of \( x(n_1, n_2) \) and a 2D fractal surface that can, at most, possess self-similarity over a magnification (zoom) factor of about 100, at which point the resolution is so coarse that 'surface' features are not visually interpreted by the human vision system.

The DFT synthesis method has additional limitations. The number of real multiplications \( N_{\text{mult}}^{(\text{FFT})} \) and real additions \( N_{\text{add}}^{(\text{FFT})} \), assuming a fast Fourier transform (FFT) implementation of the DFT is given, for the 2D case, by

\[
N_{\text{mult}}^{(\text{FFT})} = 3M_1M_2 \log_2(M_1M_2) \\
N_{\text{add}}^{(\text{FFT})} = 2M_1M_2 \log_2(M_1M_2).
\]

This often imposes a practical limit on the maximum block size that can be processed by the FFT in an acceptable period of time. The coverage of very large images \( x(n) \), say \( M_i > 10^4 \), is often only practical if \( x(n) \) is broken into smaller rectangles and by applying the FFT to each rectangular block. However, block processing causes intolerable edge effects, as shown in Fig. 1 for the 2D case, where \( M_1 = 64 \) and \( M_2 = 64 \) in each of the four rectangular blocks.

A final disadvantage of the DFT approach is that it is not possible to spatially-vary the statistics of the fractal image within each data block. The spectral exponent \( \beta \), and therefore the overall appearance of the surface features, are necessarily constant (independent of \( n \)) in each block of Fig. 1. It would be very useful, from a practical point of view, if the "roughness" of the fractal surface could be arbitrarily varied throughout the image \( x(n) \) by choosing \( \beta(n) \) to be a function of \( n \). In this way, mountains could merge gradually into smooth valleys, for example, without the requirement for postprocessing of the fractal image [2]. It might also be desirable to spatially-vary the exponent \( \beta \) as a function of angular orientation so that surfaces are "rougher" in some directions than in other directions, thereby attributing direction-oriented texture and roughness to a surface.
III. APPROXIMATION OF MD Bm USING RECURSIVE MD FILTERS EXCITED BY MD NOISE

Assume a continuous-domain MD noise source \( x(t) \) having a Gaussian amplitude probability function, zero mean and unity variance. Let the MD Fourier transform of \( x(t) \) be \( X(j\omega) \) and assume that the corresponding MD spectral density function \( \Phi_x(\omega) \) is known and that \( \omega \in \mathbb{R} \).

In practice, we require a discrete-domain version \( x(n) \) of this noise process having region of support \( Z^M \), for which the corresponding continuous-domain spectral density function \( \Phi_x(\omega) = \Phi_{x0}(\omega) = \sum_{\omega = -\infty}^{\infty} \Phi_x(\omega) \delta(\omega - \omega_n) \), \( \omega \in \mathbb{R}^M \), but has periodicity \( 2\pi \) in each frequency variable \( \omega_i \). We define the hyperrectangular region \( R^M \) of \( \mathbb{R}^M \) as

\[
R^M = \{ \omega \in \mathbb{R}^M | -\pi \leq \omega_1, \omega_2, \ldots, \omega_M \leq \pi \}
\]

and we propose to approximate the spectral density function \( \Phi(\omega) \) of (3) over this region. Consider a prototype discrete-domain MD noise signal \( x(n) \) ideally having the continuous-domain MD spectral density function \( \Phi_x(\omega) = \Phi_{x0}(\omega) = \sum_{\omega = -\infty}^{\infty} \Phi_x(\omega) \delta(\omega - \omega_n) \), \( \omega \in \mathbb{R}^M \). Let \( x(n) \) be applied to a MD digital filter having a unit impulse response \( h(n) \) and a MD hyperspherically-symmetric continuous-domain energy density spectrum \( \Phi_h(\Omega) \) given by

\[
\Phi_h(\omega) = H(j\omega)H(-j\omega) = \Phi_{h0}(||\omega||^2)^{-\beta_h}, \omega \in R^M
\]

where \( H(j\omega) \) is the Fourier transform \( F[h(n)] \) and the magnitude response \( M(\omega) \) is given by

\[
M(\omega) = |H(j\omega)| = \Phi_{h0}^{1/2}(||\omega||^2)^{-\beta_h/2}, \omega \in R^M
\]

Writing

\[
\beta = \alpha + \beta_h
\]

it follows that the MD output signal \( y(n) = (x(n) * h(n)) \) has a corresponding continuous-domain spectral density function \( \Phi_y(\omega) \) given by

\[
\Phi_y(\omega) = \Phi_x(\omega)\Phi_h(\omega) = \Phi_{y0}(||\omega||^2)^{-\beta}, \omega \in R^M
\]

where \( \Phi_{y0} \) is defined as \( \Phi_{y0} = \Phi_{x0}\Phi_{h0} \). Equation (14) corresponds to one MD period of \( \Phi_y(\omega) \) in \( R^M \) and is the required spectral density function of the discrete-domain simulated fBm signal \( y(n) \).

It is interesting to note that MD recursive filters are well suited to simulate MD fBm in this way. In general, \( 1/f \)-type noise is characterized as having long-range dependence or "memory", where current values of a process are influenced by its entire history, most strongly by recent events and by decaying amounts for increasingly distant events [5]. We employ MD recursive filters to model this notion of memory via the convolution operation \( y(n) = x(n) * h(n) \). It is important to recognize that the impulse responses \( h(n) \) of these filters are exponential in nature and can only approximate the power-law decay of the fBm.

In the remainder of this contribution, we describe a 2D application of this principle using 'white' 2D noise \( \alpha = 0 \) as the input signal \( x(n_1, n_2) \) to a 2D recursive digital filter \( h(n_1, n_2) \) that is designed to have an approximately circularly-symmetric magnitude frequency response function \( M(\omega_1, \omega_2) \).

IV. GENERATION OF 2D FRAC TAL LANDSCAPE APPROXIMATIONS USING RECURSIVE DIGITAL FILTERS

Choosing the spectral exponent \( \alpha \) of the 2D noise samples \( x(n_1, n_2) \) as zero corresponds to an easily-implemented spectrally-flat noise source. We must then implement a 2D recursive filter such that its continuous-domain circularly-symmetric magnitude frequency response is given by

\[
M(\omega_1, \omega_2) \approx \Phi_{y0}^{1/2}(\omega_1^2 + \omega_2^2)^{-\beta/2}
\]

where \( \Phi_{y0}^{1/2} \) is an arbitrary constant and \( \beta \) is the required spectral exponent of the output fBm approximating \( y(n_1, n_2) \). The frequency domain \( (\omega_1, \omega_2) \) is continuous on \( R^2 \) and \( M(\omega_1, \omega_2) \) is circularly-symmetric, periodic, and integer multiples of \( 2\pi \) on \( \omega_1 \) and \( \omega_2 \). Consequently, we shall approximate \( M(\omega) \) in (12) over the region of \( \omega \) given by

\[
R^2 = \{ \omega_1, \omega_2 \in R^2 | -\pi \leq \omega_1, \omega_2 \leq \pi \}
\]

where unit spatial sample distances are assumed; that is, \( n_1, n_2 \in Z \).

Although there is a significant body of literature relating to 2D circularly-symmetric frequency responses [7], the design of 2D filters according to (12) is non-trivial. Given the 2D discrete transfer function \( H(z_1, z_2) \) in the matrix form

\[
H(z_1, z_2) = \frac{[1 + z_1^{-1} z_2^{-2} \ldots z_1^{-K_1}] |A|}{[1 + z_1^{-1} z_2^{-2} \ldots z_1^{-K_1}] |B|} \times \frac{[1 + z_2^{-1} z_2^{-2} \ldots z_2^{-K_2}] |B|}{[1 + z_2^{-1} z_2^{-2} \ldots z_2^{-K_2}] |B|}
\]

we require a stable implementation having acceptably low orders \( K_1 \) and \( K_2 \) and a magnitude response \( M(\omega_1, \omega_2) = |H(e^{j\omega_1}, e^{j\omega_2})| \) that approximates \( M(\omega) \) of (12) in \( R^2 \).

Domain of the Approximation It is most important that \( M(\omega_1, \omega_2) \) be approximated in a satisfactory way in the region \( R^2 \). Near the origin of \( R^2 \), say where \( ||\omega|| < \pi/10 \), it is particularly important that the approximation be valid because this region of the \( \omega \) domain corresponds to the surface features that are readily detectable by the human vision system as landscape features. If the image \( y(n_1, n_2) \) has support \( Z^2 \) \( n_1, n_2 
\leq M/2 \), then it is then necessary to maintain the fBm approximation at frequencies closer to the origin than about \( ||\omega|| < \pi/M \) because the corresponding spatial-domain features have spatial-constants that exceed the size \( M_1 \times M_2 \) of the image and cannot be viewed. The accuracy of the fBm approximation is also non-critical in regions well-removed from the origin, where \( ||\omega|| > \pi/4 \), because such spectral components do not contribute significantly to overall perceived shape but, rather, to the fine texture of the perceived surface.

A. Approximation of fBm Using a 2D Recursive Digital Filter Employing Rotated 1D Filters

Consider the radial line \( L_i \) in \( R^2 \) at an angle \( \theta_i \) as shown in Fig. 2. We shall design a rotated 1D recursive digital filter \( H_i(z_1, z_2) \) having a 2D magnitude frequency response given
by some appropriate function $M_i(\gamma_i)$ where

$$\gamma_i = \omega_1 \cos \theta_i + \omega_2 \sin \theta_i,$$  \hspace{1cm} (18)

In directions parallel to $L_i$, the gain $M_i(\gamma_i)$ is therefore constant, as indicated by the gain contours shown in Fig. 2.

Consider now the set of $N$ radial lines $L_i$ shown in Fig. 3, corresponding to equi-spaced radial lines in $R^2_\theta$ given by

$$\theta_i = -\frac{\pi}{2} + \frac{\pi i}{2N}, \quad i = 0, 1, \ldots, N - 1$$  \hspace{1cm} (19)

and the corresponding rotated-1D filters $H_i(z_1, z_2)$ having the gain functions $M_i(\gamma_i)$. The serial connection of such $N$ rotated-1D filters has the transform transfer function

$$H(z_1, z_2) = \prod_{i=1}^{N} H_i(z_1, z_2)$$  \hspace{1cm} (20)

and frequency response

$$M(\omega_1, \omega_2) = \prod_{i=1}^{N} M_i(\gamma_i)$$  \hspace{1cm} (21)

Clearly, $M(\omega_1, \omega_2)$ has $2N$-side polygonal symmetry about the origin and not the circular symmetry that we are seeking. Further, we have so far not determined how to select the function $M_i$. It is shown in Appendix A that, with

$$M_i(\gamma_i) = \frac{1}{(\gamma_i)^{\delta_k}},$$  \hspace{1cm} (22)

substituting (22) into (21) gives the 2D magnitude frequency response

$$M(\omega_1, \omega_2) = \prod_{k=0}^{N-1} \frac{1}{(|\omega_1| \sin \delta_k)^{\lambda}},$$  \hspace{1cm} (23)

where the angle $\delta_k$ is shown in Fig. 3 and is given, for any line $L_i$, $i = 0, 1, 2, \ldots, N$, by

$$\delta_k = -\frac{\pi}{2} + \frac{\pi k}{N}, \quad k = 0, 1, \ldots, N - 1$$  \hspace{1cm} (24)

The magnitude response in (23) exactly matches the required circularly-symmetric magnitude response along all $N$ radial lines $L_i$ if we choose the spectral exponent $\lambda$ as

$$\lambda = \frac{\beta}{2(N - 1)}$$  \hspace{1cm} (25)

where $\beta = 3$ shown on frequency axes normalized to $\pi$.

(a) $-\pi \leq \omega_1, \omega_2 \leq \pi$.

(b) $-1.1 \leq \omega_1, \omega_2 \leq 1.1$.

The magnitude response in (23) exactly matches the required circularly-symmetric magnitude response along all $N$ radial lines $L_i$ if we choose the spectral exponent $\lambda$ as

$$\lambda = \frac{\beta}{2(N - 1)}$$  \hspace{1cm} (26)

B. 2D Numerical Example. $N = 6, \beta = 3$

We want to simulate 2D fBm in an image $y(n_1, n_2)$ having size $(M_1, M_2) = (512, 512)$ and having a fractal dimension $D = 2$; that is, according to (2) and (4), $\beta = 3$. Then, from (12), the required magnitude frequency response of the 2D filter, when driven by 2D white noise, has the form $M(\omega_1, \omega_2) = \Phi^{1/2}(\omega_1^2 + \omega_2^2)$. This is shown on a linear vertical axis in Figs. 4(a) and 4(b). The most important region of $R^2_\theta, -\pi/10 < \omega_1, \omega_2 < \pi/10$, is shown in Fig. 4(b).
Consider the choice \( N = 6 \), corresponding to the six lines \( L_1, L_2, \ldots, L_6 \) in angular increments of \( \pi/6 \), according to (19). From (26), we choose \( \lambda = 0.3 \) and use (18)–(22) to obtain the approximation of \( M(\omega_1, \omega_2) \) shown in Figs. 5(a) and 5(b) which closely approximates the required \( M(\omega_1, \omega_2) = \Phi_{0}^{1/2}(\omega_1^2 + \omega_2^2)^{-3/2} \) shown in Figs. 4(a) and 4(b). It is concluded that we may use these six rotated 1D filters to implement the required 2D filter.

V. 2D FILTER IMPLEMENTATION USING SIX ROTATED 1D FILTERS AND A SUB-SAMPLING SCHEME

Consider the six lines \( L_1, L_2, \ldots, L_6 \) shown in Fig. 6. The six 1D filters \( H_i(z_1, z_2) \) used in (20) are to be designed from a continuous-domain lowpass prototype function

\[
P(s) = \frac{[c_0 c_1 \cdots c_K][1 + s^K]^T}{[d_0 d_1 \cdots d_K][1 + s^K]^T}
\]

(27)

The discrete-domain prototype filter \( G(z) \) may then be obtained by the bilinear transformation as

\[
G(z) 
\equiv P(s)\big|_{s = \frac{z - 1}{z + 1}}
\]

\[
= \frac{[a_0 a_1 \cdots a_K][1 + z^{-1} + z^{-K}]^T}{[b_0 b_1 \cdots b_K][1 + z^{-1} + z^{-K}]^T}
\]

(28)

where \( T \) is the intersample distance on the rectangular sample grid and where \( P(s) \) is designed such that

\[
M(\omega) \equiv |G(e^{j\omega})| = M_0|\omega|^{-\lambda},
\]

(29)

as required.

The two axis-aligned filters, corresponding to \( L_1 \) and \( L_4 \), can be implemented simply as \( H_1(z_1, z_2) = G_1(z_2) \) and \( H_4(z_1, z_2) = G_2(z_2) \), respectively. Now consider \( H_2(z_1, z_2) \), corresponding to alignment on the off-axis line \( L_2 \) in Fig. 6.

The rotated function \( H_2(z_1, z_2) \) can be obtained from (27) by the conventional s-domain rotation

\[
s = s_1 \cos \theta_1 + s_2 \sin \theta_1
\]

(30)

such that

\[
H_2(z_1, z_2) = P(s_1 \cos \theta_2 + s_2 \sin \theta_2)
\]

\[
|_{s_1, s_2 = \frac{z_1, z_2}{\sqrt{1 + z_1^2 + z_2^2}}}
\]

(31)

Similarly, the 3 other off-axis filters, corresponding to \( L_i, i = 3, 5, 6 \), can be obtained as

\[
H_i(z_1, z_2) = P(s_1 \cos \theta_i + s_2 \sin \theta_i)
\]

\[
|_{s_1, s_2 = \frac{z_1, z_2}{\sqrt{1 + z_1^2 + z_2^2}}}
\]

(32)

However, we have chosen not to employ this conventional approach to the design of the s-domain-rotated 2D filters along \( L_i, i = 2, 3, 5, 6 \), but instead to use a sub-sampling technique that corresponds to 1D filters along lines \( L'_i, i = 2, 3, 5, 6 \). These lines are very close to the lines \( L_i, i = 2, 3, 5, 6 \). The implementation of 1D filters provides improved computational efficiency relative to the corresponding 2D filters obtained in (31) and (32). We note that, for \( L_2 \), \( \tan(\theta_2) = \tan(\pi/6) = 0.577 \). We choose the line \( L'_2 \) having angle \( \theta'_2 \) such that \( \tan(\theta'_2) = 0.50 \). Similarly, we choose \( L'_i \) and \( \theta'_i, i = 3, 5, 6 \), as follows:

\[
\tan(\theta'_2) = 2.0
\]

\[
\tan(\theta'_3) = -2.0
\]

\[
\tan(\theta'_5) = -0.5
\]

(33)

The lines \( L'_2, L'_6 \) exactly intersect the sub-sample points \((2n_1, n_2)\), as shown in Fig. 7, and the lines \( L'_2, L'_5 \) exactly intersect the sub-sample points \((n_1, 2n_2)\). We propose to design 1D filters \( G_i(z), i = 2, 3, 5, 6 \), that have their sample points along the lines \( L'_i, i = 2, 3, 5, 6 \), implying that their frequency responses on \( \omega_1, \omega_2 \) are rotated versions of the 1D filter response \( M(\omega) = M_0|\omega|^{-\lambda} \). The 1D filters are easily designed from \( G(z) \); we need only take into account the increased intersample distance of \( T = \sqrt{5} \) in (28). Let \( G_{\theta'_2}(z) \) denote the discrete-domain prototype function that corresponds to (28) with \( T = \sqrt{5} \). For the line \( L'_2 \), we employ the 1D filter

\[
H_2(z_1, z_2) = G_{\theta'_2}(z_1, z_2) \equiv G_2(z),
\]

where the shift operator

\[
z = z_1^2 z_2
\]

corresponds to a shift by two pixels in \( n_1 \) and one pixel in \( n_2 \), which is implemented as a 1D recursion.
along the line $L_2'$. Similarly, the remaining sub-sampled 1D filters are given by

$$
H_3(z_1, z_2) = G_{\sqrt{3}}(z_1^2 z_2^2) \equiv G_3(z)
$$
$$
H_5(z_1, z_2^2) = G_{\sqrt{5}}(z_1^{-1} z_2^2) \equiv G_5(z)
$$
$$
H_6(z_1, z_2) = G_{\sqrt{6}}(z_1^{-2} z_2) \equiv G_6(z).
$$

(34)

The final 2D transfer function of the required filter is then given by

$$
H(z_1, z_2) = \prod_{i=1}^{6} G_i(z) = G(z_1) \cdot G(z_2) \cdot G_{\sqrt{3}}(z_1^2 z_2) \cdot
$$
$$
G_{\sqrt{5}}(z_1^{-1} z_2^2) \cdot G_{\sqrt{6}}(z_1^{-2} z_2).
$$

(35)

**FRACTAL GENERATING FILTER**

A. Design of the s-Domain Prototype Filter $P(s)$

The coefficients of a third-order ($K = 3$) prototype function $P(s)$ corresponding to (27) and (29) and with $\lambda = 0.3$ have been obtained by the conventional numerical optimization of $M(\omega)$ to approximate that in (29) and are given in Table I along with the coefficients for both $G(z)$ and $G_{\sqrt{3}}(z)$. The numerically-optimized magnitude frequency response $M(\omega) = |G(e^{j\omega})|$ is shown in Fig. 8 along with the ideal response $M_0(\omega)^{-\lambda}$. Note that the zero-frequency gain $M(0)$ is tapered to unity, for practical reasons. The 2D frequency response $M(\omega_1, \omega_2) = |H(e^{j\omega_1}, e^{j\omega_2})|$, from (35), is shown in Figs. 9(a) and 9(b). This 2D frequency response closely approximates the required function $M(\omega_1, \omega_2) = \Phi_{1/2}(\omega_1^2 + \omega_2^2)^{-3/2}$ and therefore the filters in Table I are used to simulate 2D fractal landscapes. The contours of $M(\omega_1, \omega_2)$ are approximately circular and the function rolls off along the lines $L_i$ in close agreement with $\Phi_{1/2}(\omega_1^2 + \omega_2^2)^{-3/2}$, as shown in Fig. 10 for the case of $L_1, L_4$. Between these lines, the contours in Fig. 9 confirm that the approximation remains good within the required domain of approximation in $R_2^2$.

Given the order $K$ for $G(z)$, the total number of calculations for each complete 1D recursion is $(2K + 1)M_1M_2$ real multiplications and real additions. The total number of calculations for all $N$ 1D filtering operations is therefore

$$
N_{\text{add}}^{(1D)} = N(2K + 1)M_1M_2
$$
$$
N_{\text{mult}}^{(1D)} = N(2K + 1)M_1M_2.
$$

(36)

A comparison of (36) with the DFT method and (9) confirms that the proposed method is computationally more efficient for very large images (typically where $M_1, M_2 \gg 8192$). For an image of dimension $M_1 = M_2 = 16,384$, and using $N = 6$ and $K = 3$ for the prototype filter, a total of $11.274 \times 10^9$ real multiplications and additions are required. This compares to $1.032 \times 10^{10}$ real multiplications and $22.549 \times 10^9$ real additions for the 2D FFT method by (9). For smaller images, say $M_1 = M_2 = 512$, the relative computational advantage is less significant. For example, $11,010,048$ real multiplications and additions are required for the proposed method and $9,437,184$ real multiplications and $14,155,776$ real additions are required for the DFT method.

B. Example: A 2D Simulated Fractal Landscape Image with Spatially-Variant Roughness

The proposed method may be used to simulate a fractal landscape of dimension $M_1 = M_2 = 128$ having surface features that are determined by the spatially-variant spectral exponent $\beta(n_1, n_2)$, as shown in Fig. 11, from which we expect relatively smooth landscape "terrain" in the lower-left region, $\beta(n_1, n_2) = 4.0$, that becomes progressively more rugged where $\beta(n_1, n_2)$ tends to the lower limit of $3.0$. A transition zone is used to change $\beta(n_1, n_2)$ gradually and consists of 9 rings, each 4 pixels in width, as shown. $\beta(n_1, n_2)$ is decreased in increments of 0.1 through the 9 rings, from 4.0 at the interior boundary of the zone to 3.0 at the exterior boundary. Corresponding 1D prototype functions $P(s), G(z), G_{\sqrt{3}}(z)$, having $\lambda = 0.4, 0.39, 0.38, \ldots, 0.31$, respectively.
TABLE II
COEFFICIENTS OF TEN 1D PROTOTYPE FUNCTIONS HAVING $\lambda = 0.40$ TO 0.31

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 0.40$</td>
<td>$c_k$</td>
<td>-0.0023016</td>
<td>-0.034643</td>
<td>-0.042271</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0.0025777</td>
<td>0.13165</td>
<td>0.70491</td>
<td>1.0</td>
</tr>
<tr>
<td>$\lambda = 0.39$</td>
<td>$c_k$</td>
<td>-0.0024078</td>
<td>-0.036518</td>
<td>-0.043158</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0.0026954</td>
<td>0.13505</td>
<td>0.71380</td>
<td>1.0</td>
</tr>
<tr>
<td>$\lambda = 0.38$</td>
<td>$c_k$</td>
<td>-0.0024976</td>
<td>-0.03826</td>
<td>-0.044267</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0.0027938</td>
<td>0.13751</td>
<td>0.72006</td>
<td>1.0</td>
</tr>
<tr>
<td>$\lambda = 0.37$</td>
<td>$c_k$</td>
<td>-0.002601</td>
<td>-0.040198</td>
<td>-0.045253</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0.002906</td>
<td>0.14046</td>
<td>0.72759</td>
<td>1.0</td>
</tr>
<tr>
<td>$\lambda = 0.36$</td>
<td>$c_k$</td>
<td>-0.002709</td>
<td>-0.04226</td>
<td>-0.046152</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0.0030235</td>
<td>0.14357</td>
<td>0.73542</td>
<td>1.0</td>
</tr>
<tr>
<td>$\lambda = 0.35$</td>
<td>$c_k$</td>
<td>-0.0028183</td>
<td>-0.044377</td>
<td>-0.047151</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0.0031417</td>
<td>0.14654</td>
<td>0.74281</td>
<td>1.0</td>
</tr>
<tr>
<td>$\lambda = 0.34$</td>
<td>$c_k$</td>
<td>-0.0029303</td>
<td>-0.04660</td>
<td>-0.048061</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0.0032622</td>
<td>0.14955</td>
<td>0.75024</td>
<td>1.0</td>
</tr>
<tr>
<td>$\lambda = 0.33$</td>
<td>$c_k$</td>
<td>-0.0030525</td>
<td>-0.049001</td>
<td>-0.048863</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0.0033919</td>
<td>0.15285</td>
<td>0.75830</td>
<td>1.0</td>
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<tr>
<td>$\lambda = 0.32$</td>
<td>$c_k$</td>
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<tr>
<td>$d_k$</td>
<td>0.0035231</td>
<td>0.15599</td>
<td>0.76587</td>
<td>1.0</td>
</tr>
<tr>
<td>$\lambda = 0.31$</td>
<td>$c_k$</td>
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<td>-0.054095</td>
<td>-0.050545</td>
</tr>
<tr>
<td>$d_k$</td>
<td>0.0036609</td>
<td>0.15935</td>
<td>0.77389</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Fig. 9. Contour diagrams of $M(\omega_1, \omega_2)$ using (33), (a) $-\pi \leq \omega_1, \omega_2 \leq \pi$, (b) $-\pi \leq \omega_1, \omega_2 \leq \pi$.

Fig. 10. Cross-section of $M(\omega_1, \omega_2)$ in Fig. 9(a) along lines $L_1$ and $L_4$.

The 1D recursion is performed on each of the lines $L_1, L_2, L_3, L_4, L_5, L_6$, as described above. The spatial variation of $\beta(n_1, n_2)$ is achieved during the recursion on each line by simply switching to the coefficients of the appropriate $G(z)$ or $G_0(z)$ at each pixel where $\beta(n_1, n_2)$ changes value. Employing a 128 x 128 white noise source (implying $\alpha = 0$) as the input signal to the spatially-variant set of six rotated 1D filters gives the 128 x 128 fractal landscape image $y(n_1, n_2)$ shown in Fig. 12. The fractal dimension $D$ of the surface increases from the front-left corner to the far-right corner, implying progressively increasing “roughness” as the recursion proceeds away from the front-left corner.

VI. SUMMARY

In this contribution, we show that MD recursive digital filtering techniques may be employed to simulate MD random fBm. Hyperspherically-symmetric MD recursive filters may be used to filter MD input noise images that have known
Appendix A

In this appendix, we derive (23), which is the 2D frequency response along each line $L_i$ of the serial connection of $N$ rotated 1D filters. Consider two lines, $L_i$ and $L_k$, $k \neq i$, where $L_k$ is rotated from $L_i$ by an angle $\delta_k$, as shown in Fig. A1. From (22), $M_k(\gamma_k) = 1/(\gamma_k)^k$ where $\gamma_k = \omega_1 \cos \delta_k + \omega_2 \sin \delta_k$. Hence, $M_k(\gamma_k)$ has gain contours that are parallel to the line $\gamma_k = \omega_1 \cos \delta_k + \omega_2 \sin \delta_k = 0$. We wish to determine the contribution of $M(\gamma_k)$ to the overall 2D magnitude frequency response $M(\omega_1, \omega_2)$ along $L_i$. Consider a point $||\omega||_2 = (\omega_1^2 + \omega_2^2)^{1/2}$ on $L_i$, as shown in Fig. A1. The contour of $M_k(\gamma_k)$ that intersects $L_i$ at $||\omega||_2$ corresponds to $\gamma_k = ||\omega||_2 \sin \delta_k$, as shown in Fig. A1. Therefore, the 2D magnitude frequency response $M(\omega_1, \omega_2)$ along $L_i$ due to $M_k(\gamma_k)$ is

$$M(\omega_1, \omega_2) = M_k(||\omega||_2 \sin \delta_k) = \frac{1}{(||\omega||_2 \sin \delta_k)^k} \tag{A1}$$

All $M_k(\gamma_k), k \neq i$, contribute in this way to $M(\omega_1, \omega_2)$ along $L_i$, giving (23) for $\delta_k$ in (24).

References


Norman R. Bartley (S’75-M’78) received the B.Sc. and M.Sc. degrees in electrical engineering from the University of Calgary, Calgary, Alberta, Canada, in 1976 and 1978, respectively.

He is currently employed at the University of Calgary as a Research Associate. His main fields of interest include multidimensional systems, image processing, and digital signal processing systems and architectures.

Leonard T. Bruton (S’38-M’80-F’81) is a Professor of Electrical Engineering at the University of Calgary, Calgary, Alberta, Canada. His research interests are in the area of analog and digital signal processing. He is particularly interested in the design and implementation of microelectronic digital filters and the applications of multidimensional circuit and systems theory to digital image processing.